LEARNING FROM NEIGHBORS ABOUT A CHANGING STATE

KRISHNA DASARATHA, BENJAMIN GOLUB, AND NIR HAK

Abstract. Agents learn about a changing state using private signals and past actions of neighbors in a network. We characterize equilibrium learning and social influence in this setting. We then examine when agents can aggregate information well, responding quickly to recent changes. A key sufficient condition for good aggregation is that each individual's neighbors have sufficiently different types of private information. In contrast, when signals are homogeneous, aggregation is suboptimal on any network. We also examine behavioral versions of the model, and show that achieving good aggregation requires a sophisticated understanding of correlations in neighbors' actions. The model provides a Bayesian foundation for a tractable learning dynamic in networks, closely related to the DeGroot model, and offers new tools for counterfactual and welfare analyses.

Date: January 25, 2020.
Department of Economics, Harvard University, Cambridge, U.S.A., dasarath@g.harvard.edu, bgolub@fas.harvard.edu, nirhak@g.harvard.edu. We thank the Foundations for Human Behavior Initiative at Harvard for financial support. Yixi Jiang provided excellent research assistance. We are grateful to (in random order) Erik Madsen, Alireza Tahbaz-Salehi, Drew Fudenberg, Matt Elliott, Nageeb Ali, Tomasz Strzalecki, Alex Frankel, Jeroen Swinkels, Margaret Meyer, Philipp Strack, Erik Eyster, Michael Powell, Michael Ostrovsky, Leeat Yariv, Andrea Galeotti, Eric Maskin, Elliot Lipnowski, Jeff Ely, Kevin He, Eddie Dekel, Annie Liang, Iosif Pinelis, Ozan Candogan, Ariel Rubinstein, Bob Wilson, Omer Tamuz, Jeff Zwiebel, Matthew O. Jackson, Matthew Rabin, and Andrzej Skrzypacz for valuable comments, and especially to Evan Sadler for important conversations early in the project, David Hirshleifer for detailed comments on a draft, and Rohan Dutta for a helpful discussion. We also thank numerous seminar participants for helpful questions and comments.

1. Introduction

People learn from others about conditions relevant to decisions they have to make. For instance, students who are about to start their careers learn from the behavior of recent graduates. In many cases, the conditions—for example, the market returns to different specializations—are changing. Thus, welfare depends not on learning a static "state of the world" as precisely as possible, but on staying up to date with a changing state. The phenomenon of adaptation and responsiveness to new information is central in a variety of applied fields, including economic development (Griliches, 1957; Banerjee, Chandrasekhar, Duflo, and Jackson, 2013) and the study of organizations. When is a group of agents successful, collectively, in adapting efficiently to a changing environment? The answers lie partly in the structure of the social networks that shape agents' social learning opportunities. Our model is designed to analyze how a group's adaptability is shaped by the properties of such networks, the inflows of information into society, and the interplay of the two.

We consider overlapping generations of agents who are interested in tracking an unobserved *state* that evolves over time. The state is an AR(1) process, so that there is some degree of persistence in it, but also a constant arrival of innovations to learn about. Each agent, before making a decision, engages in social learning: she learns the actions of some members of prior generations, which reveal their estimates of the state.[1] The social learning opportunities are embedded in a network, in that one's network position determines the *neighborhood* of peers whom one observes. Neighborhoods reflect geographic, cultural, organizational, or other kinds of proximity. In addition to social information, agents also receive private signals about the current state, with distributions that may also vary with network position; in particular, some agents may receive more precise information about the state than others.

We give some examples. When a university student begins searching for jobs, she becomes interested in various dimensions of the relevant labor market (e.g., typical wages for someone like her), which naturally vary over time. She uses her own private research but also learns from the choices of others (e.g., recent graduates) who have recently faced a similar problem. The people she learns from are a function of her courses, dorm, and so forth: she will predominantly observe predecessors who are "nearby" in these ways.[2] Similarly, when a new cohort of professionals enters a firm (e.g., a management consultancy or law firm)

[1] In using an overlapping-generations model we follow a tradition in social learning that includes, for example, the models of Banerjee and Fudenberg (2004) and Wolitzky (2018).

[2] Sethi and Yildiz (2016) argue that, even without explicit communication costs or constraints, people can end up listening only to some others due to the investments needed to understand sources.

they learn about the business environment from their seniors. Who works with whom, and therefore who learns from whom, is shaped by the structure of the organization.

Our first contribution is to develop a network learning model suited to examples such as these. In the model, the state and the population are refreshed over time. Thus, the environment in which agents learn is dynamic, but its distribution over time is stationary. This makes equilibria and welfare simple in ways that facilitate the analysis. Indeed, our setup removes a time-dependence inherent in many models of learning, where society accumulates information over time about a fixed state: Those models imply rational updating rules that depend on the time elapsed since the "start" of the learning process. In terms of outcomes, they often focus an eventual *rate of learning* about a fixed state as in, for instance, Molavi, Tahbaz-Salehi, and Jadbabaie (2018) and Harel, Mossel, Strack, and Tamuz (2019). In contrast, our model features stationary equilibria in which learning rules are time invariant. In terms of outcomes, we focus on *steady-state learning quality* in equilibrium—how well agents track a constantly changing state.

Our second contribution is to characterize equilibria in this model and discuss the implications for some exercises of applied interest. Bayesians update estimates by taking linear combinations of neighbors' estimates and their own private information. The weights are endogenously determined because, when each agent extracts information from neighbors' estimates, the information content of those estimates depends on the neighbors' learning rules. We characterize these weights and the distributions of behavior in a stationary equilibrium.

We then discuss some implications for classic questions in networks. For instance, we define a notion of steady-state social influence—how an idiosyncratic change in an individual's information affects average behavior. This is analogous to exercises familiar from the standard DeGroot model of network learning (where the weights agents place on others are given exogenously). The endogenous determination of weights makes a big difference for how network structure maps into influence.[3] Individuals who are highly central in the social network can have low or even negative social influence once endogenous weights are taken into account. More broadly, the model permits various comparative statics, welfare analyses, counterfactuals, and estimation exercises—both under the Bayesian equilibrium benchmark and behavioral alternatives. The fact that equilibria are quite easy to compute numerically (in networks of thousands of nodes) makes the model practically useful for structural exercises. Indeed, an econometrician who observes some statistics of a social

[3]We discuss how the environment determines the right centrality measure to use, which gives a new foundation or interpretation of Katz-Bonacich centrality, analogous to the work of Ballester, Calvó-Armengol, and Zenou (2006) in network games.

learning process (such as agents' equilibrium precisions) can use the theory to estimate the underlying network and information structure.

Turning from the methods to substantive findings, our third contribution is to analyze the steady-state quality of aggregation of social information, and how it depends on signal endowments and network structure. At every point in time, agents use neighbors' actions to form an estimate of the past states that those actions reflect. Our measure of aggregation quality is the accuracy of these estimates. The main finding is that, in large Bayesian populations, an essentially optimal benchmark is achieved in an equilibrium, as long as each individual has access to a set of neighbors that is sufficiently diverse, in the sense of having different signal distributions from each other. A key mechanism behind the value of diverse signal endowments is that it leads to diversity of neighbors' strategies. This avoids "collinearity problems" in agents' sources of information, which helps them to construct better statistical estimators of the most recent state. In other words, if neighbors' strategies are diverse, an agent can more successfully filter out older, less useful information in order to concentrate on new developments in the state.

If signal endowments are *not* diverse, then our good-aggregation result does not hold. Indeed, equilibrium learning can be bounded far away from efficiency, even though each agent has access to an unbounded number of observations, each containing independent information. Thus, in stark contrast with an unchanging-state version of the model, Bayesian agents who understand the environment perfectly are not guaranteed to be able to aggregate information well. We first make this point in highly symmetric networks, where we can show the failure of aggregation is quite severe. It is natural to ask whether diversity of network positions (as opposed to signals) can make neighbors' equilibrium strategies sufficiently diverse and thus substitute for diversity of information endowments. We show that the answer is no: network asymmetry is a poor substitute for asymmetric signal distributions. In large networks, it is impossible in equilibrium to achieve accuracies of aggregation of the same order as that guaranteed in our positive result under signal diversity.

To achieve good learning when it is possible, agents must have a sophisticated understanding of the correlations in their neighbors' estimates. Thus, the second contrast we emphasize is between Bayesians who are correctly specified about their learning environment (in particular, others' behavior) and agents who do not have sufficient sophistication about the correlations among their observations to remove confounds. To make the point that such understanding is essential to good aggregation, we examine some canonical types of learning rules, adapted to our setting, in which this sophistication is absent (Eyster and

Rabin, 2010).[4] There, information aggregation is essentially guaranteed to fall short of good aggregation benchmarks for all agents. We then generalize these results to a broader class of learning rules in which agents do not use sophisticated filtering techniques. We argue that the deficiencies of naive learning rules are different from and more severe than those in similar problems with an unchanging state, and the requirements for good learning are more stringent. In analogous fixed-state environments where individuals have sufficiently many observations, if everyone uses certain simple and stationary heuristics (requiring no sophistication about correlations between neighbors' behavior), they can learn the state quite precisely. This point is made in the most closely analogous environments by Golub and Jackson (2010) and Jadbabaie, Molavi, Sandroni, and Tahbaz-Salehi (2012), where DeGroot-style rules guarantee asymptotically efficient learning. A changing state makes such imitative heuristics quite inefficient. It is worth emphasizing that even Bayesians' good learning in our environment depends on conditions—namely, signal diversity throughout the network—that differ markedly from the conditions that play a role in the above-mentioned models with an unchanging state.

Some of our theoretical aggregation results use large random graphs and make some assumptions on their distribution. We perform a numerical exercise to show that the main qualitative message—diversity of signal types helps learning—holds in real networks with a few hundred agents and average neighborhood sizes between 10 and 20. In this exercise, we use Indian village social networks from the data of Banerjee, Chandrasekhar, Duflo, and Jackson (2013) and calculate equilibria in our model for both homogeneous and non-homogeneous signal endowments. We then quantify how much better learning becomes with a certain amount of signal diversity. First, we endow all agents with exchangeable signals. In these homogeneous-signal environments, there is some variation in learning quality due to differences in networks across different villages. We compare this variation to the benefit of introducing diversity of signal endowments. Introducing a moderate amount of heterogeneity[5] in signal distributions improves outcomes by as much as moving 6.5 standard deviations in the distribution of homogeneous-signal environments. In fact, this improvement is substantially larger than the gap between the best and worst villages under homogeneous signals.

[4]See also Bala and Goyal (1998), a seminal model of boundedly rational learning rules in networks.

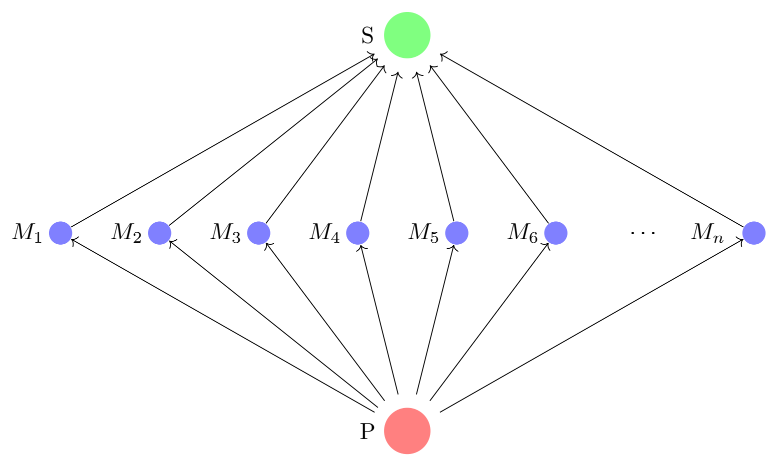
[5]We increase private signal variances of half of agents by 50% and adjust the remaining private signal distributions keeping the average precision unchanged (so that the amount of information coming into the community is held fixed).

Finally, we discuss some implications of our results for designers who wish to facilitate better learning, and what distributions of expertise they would prefer. In particular, our results provide a distinctive rationale for *informational specialization* in organizations, which we flesh out in Section 7.4. We explain there how the points we have made in a unidimensional model extend readily to richer models of multidimensional states and signals.

An Example. We now present a simple example that illustrates our dynamic model, highlights obstacles to learning that distinctively arise in a dynamic environment, and gives a sense of some of the main forces that play a role in our results on the quality of learning.

Consider a particular environment, with a single perfectly informed source S; many media outlets $M_1, \dots, M_n$ with access to the source as well as some independent private information; and the general public. The public consists of many individuals who learn only from the media outlets. We are interested in how well each member of the public could learn by following many media outlets. More precisely, we consider the example shown in Figure 1.1 and think of P as a generic member of the large public.

FIGURE 1.1. The network used in the "value of diversity" example



The state θ_t follows a Gaussian random walk: $\theta_t = \theta_{t-1} + \nu_t$, where the innovations ν_t are standard normal. Each period, the source learns the state θ_t and announces this to the media outlets. The media outlets observe the source's announcement from the previous period, which is θ_{t-1}. At each time period, they also receive noisy private signals, $s_{M_i,t} = \theta_t + \eta_{M_i,t}$ with normally distributed, independent, mean-zero errors $\eta_{M_i,t}$. They then announce their posterior means of θ_t, which we denote by $a_{M_i,t}$. The member of the public, in a given period t, makes an estimate based on the observations $a_{M_1,t-1}, \dots, a_{M_n,t-1}$ of media outlets' actions in the previous period. All agents are short-lived: they see actions in their neighborhoods one period ago, and then they take an action that reveals their posterior belief of the state.

If we instead had a fixed state but the same signals and observation structure, learning would trivially be perfect: the media outlets learn the state from the source and report it to the public.

Now consider the dynamic environment. Given that P has no signal, an upper bound on what she can hope for is to learn θ_{t-1} (and use that as her estimate of θ_t). Can this benchmark be achieved, and if so, when?

A typical estimate of a media outlet at time t is a linear combination of $s_{M_i,t}$ and θ_{t-1} (the latter being the social information that the media outlets learned from the source). In particular, the estimate can be expressed as

$$a_{M_i,t} = w_i s_{M_i,t} + (1 - w_i)\theta_{t-1},$$

where the weight w_i on the media outlet's signal is increasing in the precision of that signal. We give the public no private signal, for simplicity only.

Suppose first that the media outlets have identically distributed private signals. Because the member of the public observes many symmetric media outlets, it turns out that her best estimate of the state, $a_{P,t}$, is simply the average of the estimates of the media outlets. Since each of these outlets uses the same weight $w_i = w$ on its private signal, we may write

$$a_{P,t} = w \sum_{i=1}^{n} \frac{s_{M_i,t-1}}{n} + (1 - w)\theta_{t-2} \approx w\theta_{t-1} + (1 - w)\theta_{t-2}.$$

That is, P's estimate is an average of media private signals from last period, combined with what the media learned from the source, which tracks the state in the period before that. In the approximate equality, we have used the fact that an average of many private signals is approximately equal to the state, by our assumption of independent errors. No matter how many media outlets there are, and even though each has independent information about θ_{t-1}, the public's beliefs are confounded by older information.

What if, instead, half of the media outlets (say $M_1, \ldots, M_{n/2}$) have more precise private signals than the other half, perhaps because these outlets have invested more heavily in covering this topic? The media outlets with more precise signals will then place weight w_A on their private signals, while the media outlets with less precise signals use a smaller weight w^B. We will now argue that a member of the public can extract more information from the media in this setting. In particular, she can first compute the averages of the two groups' actions

$$w_A \sum_{i=1}^{n/2} \frac{s_{M_i,t-1}}{n/2} + (1-w_A)\theta_{t-2} \approx w_A\theta_{t-1} + (1-w_A)\theta_{t-2}$$

$$w_B \sum_{i=n/2+1}^{n} \frac{s_{M_i,t-1}}{n/2} + (1-w_B)\theta_{t-2} \approx w_B\theta_{t-1} + (1-w_B)\theta_{t-2}.$$

Then, since $w_A > w_B$, the public knows two distinct linear combinations of θ_{t-1} and θ_{t-2}. The state θ_{t-1} is identified from these. So the member of the public can form a very precise estimate of θ_{t-1}—which, recall, is as well as she can hope to do. The key force is that the two groups of media outlets give different mixes of the old information and the more recent state, and by understanding this, the public can infer both. Note that if agents are naive, e.g., if they think that all of the signals from the media are not (strongly) correlated conditional on the state, they will put positive weights on their observations and will again be bounded from learning the state.

This illustration relied on a particular network with several special features, including one-directional links and no communication among the media outlets or public. We will show that the same considerations determine learning quality in a large class of random networks in which agents have many neighbors, with complex connections among them. Quite generally, diversity of signal endowments in their neighborhoods allows agents to concentrate on new developments in the state while filtering out old, less relevant information and thus estimate the changing state as accurately as physical constraints allow.

Outline. Section 2 sets up the basic model and discusses its interpretation. Section 3 defines our equilibrium concept and shows that equilibria exist. In Section 4, we give our main theoretical results on the quality of learning and information aggregation. In Section 5, we discuss learning outcomes with naive agents and more generally without sophisticated responses to correlations. Section 6 relates our model and results to the social learning literature. In Section 7, we discuss our numerical exercise and how one could structurally estimate our model. Finally, we comment on how the main findings on the importance of signal diversity are robust to relaxing stylized assumptions of our model, such as a fixed network, unidimensional states, and Gaussian distributions.

2. Model

We describe the environment and game; complete details are formalized in Appendix A.

State of the world. There is a discrete set of instants of time,

$$\mathcal{T} = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

At each $t \in \mathcal{T}$, there is a state of the world, a random variable θ_t taking values in $\mathbb{R}$. This state evolves as an AR(1) stochastic process. That is,

$$\theta_{t+1} = \rho\theta_t + \nu_{t+1}, \tag{2.1}$$

where ρ is a constant with $0 < |\rho| \leq 1$ and $\nu_{t+1} \sim \mathcal{N}(0, \sigma_\nu^2)$ are independent innovations. We can write explicitly

$$\theta_t = \sum_{\ell=0}^{\infty} \rho^\ell \nu_{t-\ell},$$

and thus $\theta_t \sim \mathcal{N}\left(0, \frac{\sigma_\nu^2}{1-\rho^2}\right)$. We make the normalization $\sigma_\nu = 1$ throughout.

As an alternative, we will also sometimes consider a specification with a starting time, $\mathcal{T} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, where the state process can be defined as in (2.1) starting at time 0.

Information and observations. The set of *nodes* is $N = \{1, 2, \ldots, n\}$. Each node i is associated with a set $N_i \subseteq N$ of nodes that i can observe, called its *neighborhood*.[6]

Each node is populated by a sequence of *agents* in overlapping generations. At each time t, there is a node-i agent, labeled (i, t), who takes that node's action $a_{i,t}$. This agent is born at time $t - m$ at a certain location (node) and has m periods to observe the actions taken around her before she acts. Thus, when taking her action, the agent (i, t) knows $a_{j,t-\ell}$ for all nodes $j \in N_i$ and lags $\ell \in \{1, 2, \ldots, m\}$. We call m the *memory*; it reflects how many periods of neighbors' actions an agent passively observes before acting. One interpretation is that a node corresponds to a role in an organization. A worker in that role has some time to observe colleagues in related roles before choosing a once-and-for-all action herself. Much of our analysis is done for an arbitrary finite m; we view the restriction to finite memory as an assumption that avoids technical complications, but because m can be arbitrarily large, this restriction has little substantive content.[7]

Each agent also sees a private signal,

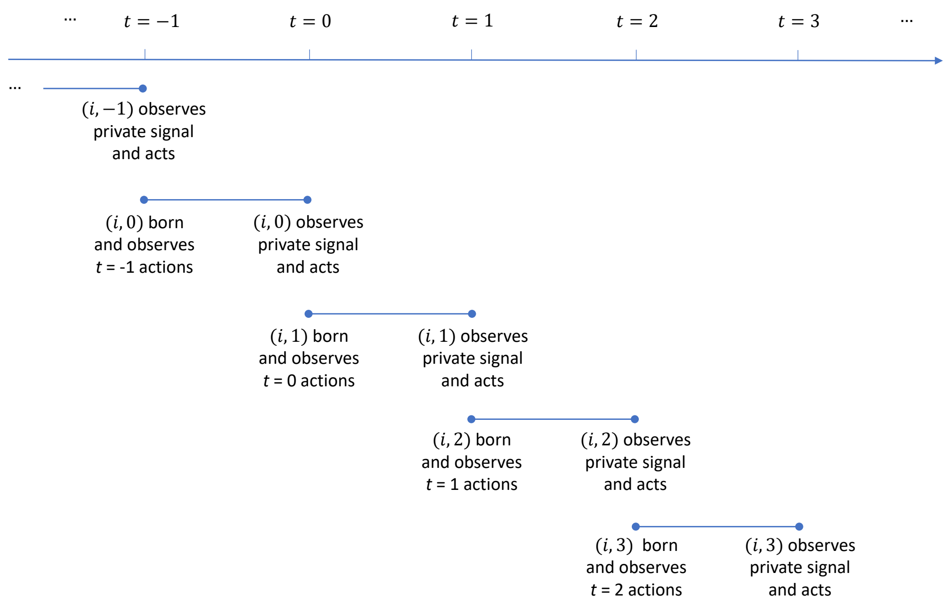
$$s_{i,t} = \theta_t + \eta_{i,t},$$

where $\eta_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$ has a variance $\sigma_i^2 > 0$ that depends on the agent but not on the time period. All the $\eta_{i,t}$ and ν_t are independent of each other. An agent's information is

[6]For all results, a node i's neighborhood can, but need not, include i itself.

[7]It is worth noting that even when the observation window m is small, observed actions can indirectly incorporate signals from much further in the past.

FIGURE 2.1. An illustration of the overlapping generations structure of the model for $m = 1$.



At time $t-1$, agent (i,t) is born and observes actions taken at time $t-1$ in her neighborhood. Then, at time t, she observes her private signal $s_{i,t}$ and takes her action $a_{i,t}$.

a vector consisting of her private signal and all of her social observations. An important special case will be $m = 1$, where agents observe only one period of others' behavior before acting themselves, so that the agent's information is $(s_{i,t}, (a_{j,t-1})_{j \in N_i})$.

The observation structure is common knowledge, as is the informational environment (i.e., all precisions, etc.). The network $G = (N, E)$ is the set of nodes N together with the set of *links* E, defined as the subset of pairs $(i, j) \in N \times N$ such that $j \in N_i$.[8]

An *environment* is specified by $(G, \boldsymbol{\sigma})$,where $\boldsymbol{\sigma} = (\sigma_i)_{i \in N}$ is the profile of signal variances.

Preferences and best responses. As stated above, in each period t, each agent (i,t) makes her once-and-for-all choice $a_{i,t} \in \mathbb{R}$ seeking to make this action close to the current state. Utility is given by

$$u_{i,t}(a_{i,t}) = -\mathbb{E}[(a_{i,t} - \theta_t)^2].$$

[8]The links are fixed over time. It may be that each generation draws its own links; we discuss this type of extension in Section 7.3.

By a standard fact about squared-error loss functions, given the distribution of $(\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^{m}$, she sets

$$a_{i,t} = \mathbb{E}[\theta_t \mid s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^{m}]. \tag{2.2}$$

Here the notation $\boldsymbol{a}_{N_i,t}$ refers to the vector $(a_{j,t})_{j\in N_i}$. An action can be interpreted as an agent's estimate of the state, and we will sometimes use this terminology.

The conditional expectation (2.2) depends, of course, on the prior of agent (i,t) about θ_t, which, under correctly specified beliefs, has distribution $\theta_t \sim \mathcal{N}\left(0, \frac{\sigma_\nu^2}{1-\rho^2}\right)$. We actually allow the prior to be any normal distribution or a uniform improper prior.[9] It saves on notation to analyze the case where all agents have improper priors. Because actions under a normal prior are related to actions under the improper prior by a simple linear bijection—and thus have the same information content for other agents—all results immediately extend to the general case.

We discuss extensions of the basic model in various directions in Section 7.

3. Updating and equilibrium

In this section we study agents' learning behavior and present a notion of stationary equilibrium. We begin with the canonical case of Bayesian agents with correct models of others' behavior; we study other behavioral assumptions in Section 5 below.

3.1. Best-response behavior.

The first step is to analyze best-response updating behavior given others' strategies. The following observations apply whether the agents are in equilibrium or not.

A strategy of an agent is *linear* if the action taken is a linear function of the variables in her information set. We first discuss agents' best responses to linear strategies, showing that they are linear and computing them explicitly.

Suppose predecessors have played linear strategies up to time t.[10] Then we can express each action up until time t as a weighted summation of past signals. Because all innovations ν_t and signal errors $\eta_{i,t}$ are independent and Gaussian, the joint distribution of $(a_{i,t-\ell'} - \theta_t)_{i\in N,\ell'\geq 1}$ is multivariate Gaussian. It follows that $\mathbb{E}[\theta_t \mid s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^{m}]$ is a linear function of $s_{i,t}$ and $(\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^{m}$. The rest of this subsection explicitly analyzes this conditional expectation.

[9]We take priors, like the information structure and network, to be common knowledge.

[10]We will discuss this below in the context of our equilibrium concept; but one immediate motivation is that, in the model with a starting time, where $\mathcal{T} = \mathbb{Z}_{\geq 0}$, Bayesian agents' updating at $t = 0$ is a single-agent problem where optimal behavior is a a linear function of own signals only, and thus the hypothesis holds. At later times it holds by induction.

3.1.1. *Covariance matrices.* The optimal weights for an agent to place on her sources of information depend on the variances and covariances of these sources. Given a linear strategy profile played up until time t, let $\boldsymbol{V}_t$ be the $nm \times nm$ covariance matrix of the vector $(\rho^\ell a_{i,t-\ell} - \theta_t)_{i \in N,\, 0 \le \ell \le m-1}$. The entries of this vector are the differences between the best predictors of θ_t based on actions $a_{i,t-\ell}$ during the past m periods and the current state of the world. (In the case $m = 1$, this is simply the covariance matrix $\boldsymbol{V}_t = \mathrm{Cov}\left((a_{i,t} - \theta_t)_{i \in N}\right)$.) The matrix $\boldsymbol{V}_t$ records covariances of action errors: diagonal entries measure the accuracy of each action, while off-diagonal entries indicate how correlated the two agents' action errors are. The entries of $\boldsymbol{V}_t$ are denoted by $V_{ij,t}$.

3.1.2. *Best-response weights.* A strategy profile is a best-response if the weights each agent places on the variables in her information set minimize her posterior variance. We now characterize such weights in terms of the covariance matrices we have defined. Consider an agent at time t, and suppose some linear strategy profile has been played up until time t. Let $\boldsymbol{V}_{N_i,t-1}$ be a sub-matrix of $\boldsymbol{V}_{t-1}$ that contains only the rows and columns corresponding to neighbors of i[11] and let

$$\boldsymbol{C}_{i,t-1} = \begin{pmatrix} & & & 0 \\ & \boldsymbol{V}_{N_i,t-1} & & 0 \\ & & & \vdots \\ 0 & 0 & \ldots & \sigma_i^2 \end{pmatrix}.$$

Conditional on observations $(\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m$ and $s_{i,t}$, the state θ_t is normally distributed with mean

$$\underbrace{\frac{\mathbf{1}^T \boldsymbol{C}_{i,t-1}^{-1}}{\mathbf{1}^T \boldsymbol{C}_{i,t-1}^{-1} \mathbf{1}}}_{\text{agent } i\text{'s weights}} \cdot \underbrace{\begin{pmatrix} \rho \boldsymbol{a}_{N_i,t-1} \\ \vdots \\ \rho^m \boldsymbol{a}_{N_i,t-m} \\ s_{i,t} \end{pmatrix}}_{\text{agent } i\text{'s observations}}. \tag{3.1}$$

(see Example 4.4 of Kay (1993)). This gives $\mathbb{E}[\theta_t \mid s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m]$ (recall that this is the $a_{i,t}$ the agent will play). Expression (3.1) is a linear combination of the agent's signal and the observed actions; the weights in this linear combination depend on the matrix $\boldsymbol{V}_{t-1}$ (but not on realizations of any random variables). In (3.1) we use our assumption of an improper prior.[12]

[11]Explicitly, $\boldsymbol{V}_{N_i,t-1}$ are the covariances of $(\rho^\ell a_{j,t-\ell} - \theta_t)$ for all $j \in N_i$ and $\ell \in \{1, \ldots, m\}$.

[12]As we have mentioned, this is for convenience and without loss of generality. Our analysis applies equally to any proper normal prior for θ_t: To get an agent's estimate of θ_t, the formula in (3.1) would simply

We denote by $(\boldsymbol{W}_t, \boldsymbol{w}_t^s)$ a *weight profile* in period t, with $\boldsymbol{w}_t^s \in \mathbb{R}^n$ being the weights agents place on their private signals and $\boldsymbol{W}_t$ recording the weights they place on their other information.

3.1.3. *The evolution of covariance matrices under best-response behavior.* Assuming agents best-respond according to the optimal weights just described in (3.1), we can compute the resulting next-period covariance matrix $\boldsymbol{V}_t$ from the previous covariance matrix. This defines a map $\Phi : \mathcal{V} \to \mathcal{V}$, given by

$$\Phi : \boldsymbol{V}_{t-1} \mapsto \boldsymbol{V}_t. \tag{3.2}$$

This map gives the basic dynamics of the model: how a variance-covariance matrix $\boldsymbol{V}_{t-1}$ maps to a new one when all agents best-respond to an arbitrary $\boldsymbol{V}_{t-1}$. The variance-covariance matrix $\boldsymbol{V}_{t-1}$ (along with parameters of the model) determines (i) the weights agents place on their observations and (ii) the distributions of the random variables that are being combined. This yields the deterministic updating dynamic Φ. A consequence is that the weights agents place on observations are (commonly) known, and do not depend on any random realizations.

Example 1. We compute the map Φ explicitly in the case $m = 1$. We refer to the weight agent i places on $a_{j,t-1}$ (agent j's action yesterday) as $W_{ij,t}$ and the weight on $s_{i,t}$, her private signal, as $w_{i,t}^s$. Note we have, from (3.1) above, explicit expressions for these weights. Then

$$[\Phi(\boldsymbol{V})]_{ii} = (w_i^s)^2\sigma_i^2 + \sum W_{ik}W_{ik'}(\rho^2 V_{kk'} + 1) \quad \text{and} \quad [\Phi(\boldsymbol{V})]_{ij} = \sum W_{ik}W_{jk'}(\rho^2 V_{kk'} + 1). \tag{3.3}$$

3.2. **Stationary equilibrium in linear strategies.** We will now turn our attention to *stationary equilibria in linear strategies*—ones in which all agents' strategies are linear with time-invariant coefficients—though, of course, we will allow agents to consider deviating at each time to arbitrary strategies, including non-linear ones. Once we establish the existence of such equilibria, we will use the word *equilibrium* to refer to one of these unless otherwise noted.

A reason for focusing on equilibria in linear strategies comes from noting that, in the variant of the model with a starting time (i.e., the case $\mathcal{T} = \mathbb{Z}_{\geq 0}$) agents begin by using only private signals, and they do this linearly. After that, inductively applying the reasoning of Section 3.1, best-responses are linear at all future times. Taking time to extend infinitely backward is an idealization that allows us to focus on exactly stationary behavior.

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be averaged with a constant term accounting for the prior, and everyone could invert this deterministic operation to recover the same information from others' actions.

We now show the existence of stationary equilibria in linear strategies.[13]

Proposition 1. *A stationary equilibrium in linear strategies exists, and is associated with a covariance matrix* $\widehat{\boldsymbol{V}}$ *such that* $\Phi(\widehat{\boldsymbol{V}}) = \widehat{\boldsymbol{V}}$.

The proof appears in Appendix B.

At such an equilibrium, the covariance matrix $\boldsymbol{V}_t$ and all agent strategies are time-invariant. Actions are linear combinations of observations with stationary weights (which we refer to as $\widehat{W}_{ij}$ and $\widehat{w}_i^s$). The form of these rules has some resemblance to static equilibrium notions studied in the rational expectations literature (e.g., Vives, 1993; Babus and Kondor, 2018; Lambert, Ostrovsky, and Panov, 2018; Mossel, Mueller-Frank, Sly, and Tamuz, 2018), but here we explicitly examine a dynamic environment in which these emerge as steady states.

3.2.1. *Intuition for the proposition.* The idea of the argument is as follows. The goal is to apply the Brouwer fixed-point theorem to show there is a covariance matrix $\widehat{\boldsymbol{V}}$ that remains unchanged under updating. To find a convex, compact set to which we can apply the fixed-point theorem, we use the fact that when agents best respond to any beliefs about prior actions, all variances are bounded above and bounded away from zero below. This is because all agents' actions must be at least as precise in estimating θ_t as their private signals, and cannot be more precise than estimates given perfect knowledge of θ_{t-1} combined with the private signal. Because the Cauchy-Schwartz inequality bounds covariances in terms of the corresponding variances, it follows that there is a compact, convex set containing the image of Φ. This and the continuity of Φ allow us to apply the Brouwer fixed-point theorem.

In the case of $m = 1$, we can use the formula of Example 1, equation (3.3), to write the fixed-point condition $\Phi(\widehat{\boldsymbol{V}}) = \widehat{\boldsymbol{V}}$ explicitly. More generally, for any m, we can obtain a formula in terms of $\widehat{\boldsymbol{V}}$ for the weights $\widehat{W}_{ij}$ and $\widehat{w}_i^s$ in the best response to $\widehat{\boldsymbol{V}}$, in order to describe the equilibrium $\widehat{V}_{ij}$ as solving a system of polynomial equations. These equations have large degree and cannot be solved analytically except in very simple cases, but they can readily be used to solve for equilibria numerically.

The main insight is that we can find equilibria by studying action covariances; this idea applies equally to many extensions of our model. We give two examples: (1) We assume that agents observe neighbors perfectly, but one could define other observation structures. For instance, agents could observe actions with noise, or they could observe some set of

[13]Because time in this game is doubly infinite, there are some subtleties in definitions, which are dealt with in Appendix A.

linear combinations of neighbors' actions with noise. (2) We assume agents are Bayesian and best-respond rationally to the distribution of actions, but the same proof would also show that equilibria exist under other behavioral rules (see Section 5.1).[14]

3.2.2. *Social influence.* A consequence of the simple linear structure of updating rules at equilibrium is that we can compute measures of social influence. This is analogous to a basic exercise in the DeGroot model, where social influence is characterized in terms of network centrality. We define the *social influence* of i to be the total weight that all actions place on agent i's private signal in a given period t. The social influence measures the total increase in actions if i's private signal $s_{i,t}$ increases by one (due to an idiosyncratic shock, say). At equilibrium, the social influence of i is:

$$SI(i) = \sum_{j \in N} \sum_{k=0}^{\infty} (\rho \widehat{W})^k_{ji} \widehat{w}^s_i.$$

Agent i's social influence depends on the weight $\widehat{w}^s_i$ she places on her own signal as well as the weights agents place on each others' actions.

Our expression for social influence is a version of Katz-Bonacich centrality with respect to the matrix $\widehat{W}$ of weights. The decay parameter is the persistence ρ of the of the AR(1) state process.[15]

We will see in Section 4 that weights on neighbors' actions can be negative at equilibrium, as can social influence. The next result shows that the summation is nevertheless guaranteed to converge at equilibrium, which makes social influence well-defined.

Corollary 1. *The social influence $SI(i)$ is well-defined at any equilibrium and is equal to $[\mathbf{1}'(I - \rho \hat{W})^{-1}]_i \hat{w}^s_i$.*

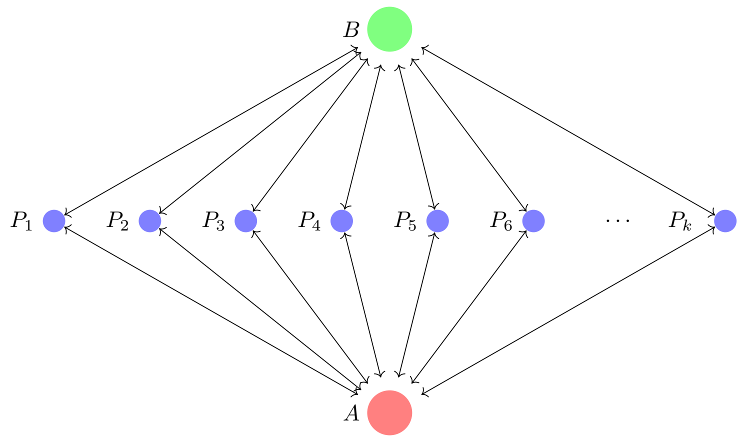
We show this as follows: if social influence did not converge, some agents would have actions with very large variances. But then these agents would have simple deviations that would improve their accuracy, such as following their private signals.

Example 2. We illustrate the computation of social influence in a simple example. There are k households, $P_1, \ldots, P_k$, who constitute the "public," and two sources, A and B. There is a bilateral link between each household and each source, and no other links. Consider

[14]What is important in the proof is that actions depend continuously on the covariance structure of an agent's observations; the action variances are uniformly bounded under the rule agents play; and there is a decaying dependence of behavior on the very distant past.

[15]This can be compared with Ballester, Calvó-Armengol, and Zenou (2006), where the decay parameter is determined by the strength of strategic complementarities in a network game. In both cases, the economic environment gives the "right" way to select this parameter.

FIGURE 3.1. The network for which we compute social influences in Example 2.



a case where all households in the public have private signal variance 1. We denote by σ_A^2 and σ_B^2 the private signal variances of the two sources.

When these variances are $\sigma_A^2 = \sigma_B^2 = 1$, the same as those of the public, the sources have social influence considerably higher than that of any member of the public because of their prominent position. However, if $\sigma_A^2 = 1$, and $\sigma_B^2 = 3$, source B's social influence can be lower than that of a member of the public, despite her much larger number of links.[16]

3.2.3. *Other remarks.* Proposition 1 shows that there exists a stationary linear equilibrium. We show later, as part of Proposition 2, that there is a unique stationary linear equilibrium in networks having a particular structure. In general, uniqueness of the equilibrium is an open question that we leave for future work.[17] In Section 4.2.3 and Appendix D we study a nonstationary variant of the model which has a unique equilibrium, and relate it to our main model.

How much information does each agent need to play her equilibrium strategy? In a stationary equilibrium, she only needs to know the steady-state variance-covariance matrix $\widehat{\boldsymbol{V}}_{N_i}$ in her neighborhood. Then her problem of inferring θ_{t-1} becomes essentially a linear regression problem. If historical empirical data on neighbors' error variances and covariances are available (i.e., the entries of the matrix $\boldsymbol{V}_{N_i,t}$ discussed in Section 3.1), then $\widehat{\boldsymbol{V}}_{N_i}$ can be estimated from this.

[16]When $\rho = 0.9$ and $k = 10$, the social influence of B is 0.65, while the typical member of the public has an influence of 0.78.

[17]We have checked numerically that Φ is not, in general, a contraction in any of the usual norms (entrywise sup, Euclidean operator norm, etc.). In computing equilibria numerically for many examples, we have not been able to find a case of equilibrium multiplicity. Indeed, in all of our numerical examples repeatedly applying Φ to an initial covariance matrix gives the same fixed point for any starting conditions.

4. How good is information aggregation in equilibrium?

In this section we define and analyze the quality of information aggregation in stationary equilibrium.

We begin with a definition. Recall that an agent at time t uses social information to form a belief about θ_{t-1}, which is a sufficient statistic for the past in the agent's decision problem. (We discuss the case $m = 1$ for simplicity but the reasoning extends easily to other values of m.) The conditional expectation of θ_{t-1} that an agent (i,t) forms is called her *social signal* and denoted by $r_{i,t}$:

$$r_{i,t} = \mathbb{E}[\theta_{t-1} \mid (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^{m}].$$

For a given strategy profile, define the *aggregation error* $\kappa_{i,t}^2 = \mathrm{Var}(r_{i,t} - \theta_{t-1})$ to be the variance of the social signal (equivalently, the expected squared error in the social signal as a prediction of θ_{t-1}). This measures how precisely an agent can extract information from social observations about the fundamental they reflect. Note that agent i's aggregation error is a monotone transformation of her expected utility.[18] We will be interested in this number, and how low error can be in equilibrium.

The environment features informational externalities: players do not internalize the impact that their learning rules have on others' learning. Consequently, there is no reason to expect aggregation error to reach efficient levels in any exact sense. And we have seen that the details of equilibrium in a particular network can be complicated. However, it turns out much more can be said about the behavior of aggregation errors as neighborhood size (i.e., the number of social observations) grows. In this section, we study the asymptotic efficiency of information aggregation. We give conditions under which aggregation error decays as quickly as physically possible, and different conditions under which it remains far from efficient levels even when agents have arbitrarily many observations.

A bechmark lower bound on aggregation error. A first observation is a lower bound on the rate at which aggregation error can decay (as a function of a node's degree, d_i) under *any* behavior of agents. This establishes a benchmark relative to which we can assess equilibrium outcomes.

Recall that the aggregation error $\kappa_{i,t}^2 = \mathrm{Var}(r_{i,t} - \theta_{t-1})$ is the variance (squared error) of the social signal.

[18] In fact, for any decision dependent on θ_t, an agent is better off with a lower value of $\kappa_{i,t}^2$. This is a consequence of the fact that unidimensional Gaussian signals can be Blackwell ordered by their precision.

Fact 1. *Fix $\rho \in (-1,1)$ as well as upper and lower bounds for private signal variances, so that $\sigma_{i,t}^2 \in [\underline{\sigma}^2, \overline{\sigma}^2]$ for all i and t. On any network and for all strategy profiles, we have $\kappa_{i,t}^2 \geq c/d_i$ for all i and t, where c is a constant that depends only on $\rho, \underline{\sigma}^2$, and $\overline{\sigma}^2$.*

The lower bound is reminiscent of the central limit theorem: if an agent had d_i conditionally independent noisy signals about θ_{t-1} (e.g., by observing neighbors' private signals directly), then the variance of her estimate would be of order $1/d_i$. Fact 1 formalizes the intuitive point that it is not possible for aggregation errors to decay (as a function of degree) any faster than that.

For an intuition, imagine that an agent sees neighbors' private signals (not just actions) one period after they are received, and all other information two periods after it was received; this clearly gives an upper bound on the quality of aggregation given physical communication constraints. The information that is two periods old cannot be very informative about θ_{t-1} because of the movement in the state from period $t-2$ to $t-1$; a large constant number z of signals about θ_{t-1} would be better. Thus, a lower bound on aggregation error is given by the error that could be achieved with $d_i + z$ independent signals about θ_{t-1} of the best possible precision ($\overline{\sigma}^2$). The bound follows from these observations.

Outline of results: When is aggregation comparable to the benchmark? Fact 1 places a lower bound on aggregation error given the physical constraints. Even efficient learning could not do better than this bound. We will examine when equilibrium learning can achieve similar-quality aggregation. More precisely, we ask when there is a stationary equilibrium where the equilibrium aggregation error at node i satisfies $\widehat{\kappa}_i^2 \leq C/d_i$ for all i, for some constant C.

In Section 4.2 we establish a good-aggregation result: outcomes comparable to the benchmark are achieved in a class of networks. The key condition enabling the asymptotically efficient equilibrium outcome described in the previous paragraph is called *signal diversity*: each individual has access to enough neighbors with multiple different kinds of private signals. The fact that neighbors use private information differently turns out to give the agents enough power to identify θ_{t-1} with equilibrium aggregation error that decays at a rate matching the lower bound up to a constant.

In Section 4.3, we turn to negative results. Without signal diversity, equilibrium aggregation can be extremely bad. Our first negative result shows that when signals are exchangeable, it may be that $\widehat{\kappa}_i^2$ does not approach zero in any equilibrium no matter how large neighborhoods are. We prove this in highly symmetric networks. Once we move away from such networks, one might ask whether diversity in individuals' network positions could play a role analogous to signal diversity and enable approximately efficient learning. Our second negative result shows that this is impossible. When signals are homogeneous, in any equilibrium, aggregation errors cannot vanish at rate C/n for any $C > 0$ as the network grows.

4.1. **Distributions of networks and signals.** For our good-aggregation result, we study large populations and specify two aspects of the environment: *network distributions* and *signal distributions*. In terms of network distributions, we work with a standard type of random network model—a stochastic block model (see, e.g., Holland et al., 1983). It makes the structure of equilibrium tractable while also allowing us to capture rich heterogeneity in network positions. We also specify *signal distributions*: how signal precisions are allocated to agents, in a way that may depend on network position. We now describe these two primitives of the model and state the assumptions we work with.

Fix a set of *network types* $k \in \mathcal{K} = \{1, 2, \ldots, K\}$. There is a probability $p_{kk'}$ for each pair of network types, which is the probability that an agent of network type k has a link to a given agent of network type k'. An assumption we maintain on these probabilities is that each network type k observes at least one network type (possibly k itself) with positive probability. There is also a vector $(\alpha_1, \ldots, \alpha_K)$ of *population shares* of each type, which we assume are all positive. Jointly, $(p_{kk'})_{k,k' \in \mathcal{K}}$ and $\boldsymbol{\alpha}$ specify the network distribution. These parameters can encode differences in expected degree and also features such as homophily (where some groups of types are linked to each other more densely than to others).

We next define signal distributions, which describe the allocation of signal variances to network types. Fix a finite set $\mathbb{S}$ of private signal variances, which we call signal types.[19] We let $q_{k\tau}$ be the share of agents of network type k with signal type τ; $(q_{k\tau})_{k \in \mathcal{K}, \tau \in \mathbb{S}}$ defines the signal distribution.

Let the nodes in network n be a disjoint union of sets $N_n^1, N_n^2, \ldots, N_n^K$, with the cardinality $|N_n^k|$ equal to $\lfloor \alpha_k n \rfloor$ or $\lceil \alpha_k n \rceil$ (rounding so that there are n agents in the network). We (deterministically) set the signal variances σ_i^2 equal to elements of $\mathbb{S}$ in accordance with the signal shares (again rounding as needed). Let $(G_n)_{n=1}^{\infty}$ be a sequence of directed or undirected random networks with these nodes, so that $i \in N_n^k$ and $j \in N_n^{k'}$ are linked with probability $p_{kk'}$; these realizations are all independent.

A *stochastic block model* D is specified by the linking probabilities $(p_{kk'})_{k,k' \in \mathcal{K}}$, the type shares $\boldsymbol{\alpha}$, and the signal distribution $(q_{k\tau})_{k \in \mathcal{K}, \tau \in \mathbb{S}}$. We let $(G_n(D), \boldsymbol{\sigma}_n(D))$ denote a realization of the network and signals under a given stochastic block model. We say that a signal type τ is represented in a network type k if $q_{k\tau} > 0$.

[19] The assumptions of finitely many signal types and network types are purely technical, and could be relaxed.

Definition 1. We say that a stochastic block model satisfies *signal diversity* if at least two distinct signal types are represented in each network type.

We will discuss stochastic block models that satisfy this condition as well as ones that do not, and show that the condition is pivotal for information-aggregation.

4.2. **Good aggregation under diverse signals.** Our first main result is that signal diversity is sufficient for good aggregation in the networks described in the previous section. Aggregation error that decays at a rate C/d_i is achieved independently of the structural properties of the network.

We first define a notion of good aggregation for an agent in terms of a bound on that agent's aggregation error.

Definition 2. Given $\epsilon > 0$, we say that agent i achieves the ϵ-aggregation benchmark if $\widehat{\kappa}_i^2 \leq \epsilon$.

We say an event occurs *asymptotically almost surely* if for any $\varepsilon > 0$, the event occurs with probability at least $1 - \varepsilon$ for n sufficiently large.

Theorem 1. *Fix any stochastic block model D. There exists $C > 0$ such that asymptotically almost surely $(G_n(D), \boldsymbol{\sigma}_n(D))$ has an equilibrium where the C/n-aggregation benchmark is achieved by all agents.*

So for large enough n, society is very likely to aggregate information very well. The uncertainty in this statement is over the network, as there is always a small probability of a realized network which prevents learning (e.g., an agent has no neighbors). We give an outline of the argument next, and the proof appears in Appendix C.

The constant C in the theorem statement can depend on D. By continuity, given any compact set of stochastic block models D, we can choose a single $C > 0$ for which the result holds uniformly across D. Thus, the theorem can be applied without detailed information on how the random graphs are generated, as long as some bounds are known about which models are possible.

4.2.1. *Discussion of the proof.* To give intuition for Theorem 1, we first describe why the theorem holds on the complete network[20] with two signal types A and B in the $m = 1$ case. This echoes the intuition of the example in the introduction. We then discuss the challenges involved in generalizing the result to our general stochastic block model networks, and the techniques we use to overcome those challenges.

[20]Note this is a special case of the stochastic block model.

Consider a time-t agent, (i,t). We define her *social signal* $r_{i,t}$ to be the optimal estimate of θ_{t-1} based on the actions she has observed in her neighborhood. On the complete network, all players have the same social signal, which we call r_t.[21]

At any equilibrium, each agent's action is a weighted average of her private signal and this social signal:[22]

$$a_{i,t} = \widehat{w}_i^s s_{i,t} + (1-\widehat{w}_i^s) r_t. \tag{4.1}$$

The weight $\widehat{w}_i^s$ depends only on the precision of agent i's signal. We call the weights used by agents of the two distinct signal types $\widehat{w}_A^s$ and $\widehat{w}_B^s$. Suppose signal type A is less accurate than signal type B, so that $\widehat{w}_A^s < \widehat{w}_B^s$.

Now observe that each time-$(t+1)$ agent can compute two averages of the time-t actions—one for each type—which can be written as follows using (4.1) and $s_{i,t} = \theta_t + \eta_{i,t}$:

$$\frac{1}{n_A} \sum_{i:\sigma_i^2 = \sigma_A^2} a_{i,t} = \widehat{w}_A^s \theta_t + (1-\widehat{w}_A^s) r_t + O(n^{-1/2}),$$
$$\frac{1}{n_B} \sum_{i:\sigma_i^2 = \sigma_B^2} a_{i,t} = \widehat{w}_B^s \theta_t + (1-\widehat{w}_B^s) r_t + O(n^{-1/2}).$$

Here n_A and n_B denote the numbers of agents of each type, and the $O(n^{-1/2})$ error terms come from the average signal noises $\eta_{i,t}$ of agents in each group, which we bound using the central limit theorem. In other words, by the law of large numbers, each time-$(t+1)$ agent can obtain precise estimates of two different convex combinations of θ_t and r_t. Because the two weights, $\widehat{w}_A^s$ and $\widehat{w}_B^s$, are distinct, she can approximately solve for θ_t as a linear combination of the average actions from each type (up to signal error). It follows that in the equilibrium we are considering, the agent must have an estimate at least as precise as what she can obtain by the strategy we have described, and will thus be very close the benchmark. The estimator of θ_t that this strategy gives will place negative weight on $\frac{1}{n_A}\sum_{i:\sigma_i^2=\sigma_A^2} a_{i,t-1}$, thus *anti-imitating* the agents of signal type A. It can be shown that the equilibrium we construct in which agents learn will also have agents anti-imitating others.

To use the same approach in general, we need to show that each individual observes a large number of neighbors of each signal type with similar social signals. More precisely, the proof shows that agents with the same network type have highly correlated social signals. Showing this is much more subtle than it was in the above illustration. In general, the

[21]In particular, agent (i,t) sees everyone's past action, including $a_{i,t-1}$.

[22]Agent i's weights on her observations $s_{i,t}$ and $\rho a_{j,t-1}$ sum to 1, because the optimal action is an unbiased estimate of θ_t.

social signals in an arbitrary network realization are enogenous objects that depend to some extent on all the links.

A key insight allowing us to overcome this difficulty is that, despite the fact that link realizations involve a lot of idiosyncratic heterogeneity, the number of paths of some length L between any two agents is nearly determined by their types, with a small relative error. In the particular asymptotic framework we have presented, this length is $L = 2$; the argument extends to other models, with a different value of L, as we discuss below. This fact allows us to study the workings of the updating map Φ (recall Section 3.1.3) in the realized random network and derive some important properties of the evolution of social signals. In particular, if we look at the set of covariance matrices where all social signals are close to perfect, we can show that Φ^L maps this set to itself (generalizing a very simple manifestation of this seen in the example above). A fixed-point theorem then implies there is a fixed point of Φ^L in this set, and further analysis allows us to deduce that there is in fact an equilibrium (corresponding to a fixed point of Φ) with nearly perfect aggregation.

We give a little more detail on how we carry this out. First, we define a high-probability *no large deviations* event, where realized L-step path counts are close in relative terms to their expectations (conditional on types). Even conditioning on this event, the differences between realized path counts and their expectations can be large in absolute terms. This can matter because the weights agents use in their updating—and thus the evolution of social signals—depend on realized network structure in a nonlinear way. It is involved to show that L-step updating is actually close enough to its expected behavior to establish what we said in the last paragraph—that good aggregation is stable under updating. A key step is to develop results on matrix perturbations to show that small relative changes in the network do not affect Φ^L too much.

4.2.2. *Sparser random graphs.* In the random graphs we have analyzed, the group-level linking probabilities $(p_{kk'})$ are, for simplicity, held fixed as n grows. This yields expected degrees that grow linearly in the population size, which may not be the desired asymptotic model. While it is important to have neighborhoods "large enough" (i.e., growing in n) to permit the application of laws of large numbers, their rate of growth can be considerably slower than linear: for example, our proof extends directly to degrees that scale as n^α for any $\alpha > 0$. Instead of studying Φ^2 and second-order neighborhoods, we apply the same analysis[23] to show that, asymptotically almost surely, there exists an equilibrium where the C/n^α-aggregation benchmark is achieved for all agents.

[23]We study Φ^L and L^{th}-order neighborhoods for L larger than $1/\alpha$.

4.2.3. *The good-aggregation outcome as a unique prediction.* The theorem above says good aggregation is supported in an equilibrium but does not state that this is the unique equilibrium outcome. To deal with this issue, we study the alternative model with $\mathcal{T} = \mathbb{Z}_{\geq 0}$ (where agents begin with only their own signals and then best-respond to the previous distribution of behavior at each time). We show that its long-run outcomes get arbitrarily close to the good-aggregation equilibrium of Theorem 1 as $n \to \infty$. Thus, even if there were other equilibria of the stationary model, they could not be reached via the natural iterative procedure coming from the $\mathcal{T} = \mathbb{Z}_{\geq 0}$ model.

We summarize the results here and defer the details to Appendix D. In the model with a starting time, there is generally a unique equilibrium outcome.[24] The counterpart of Theorem 1 says that under the same conditions on the environment as in that theorem, learning as good as the C/n-benchmark for some $C > 0$ obtains for all times $t \geq 1$. The two models predict not only similar aggregation quality but similar behavior: We also show that for any $t \geq 1$, with high probability as $n \to \infty$ the variance-covariance matrix $\boldsymbol{V}_t(n)$ becomes arbitrarily close to $\widehat{\boldsymbol{V}}(n)$, where $(\widehat{\boldsymbol{V}}(n))_n$ is any sequence of covariance matrices supporting good aggregation in the stationary model (guaranteed to exist by Theorem 1).

4.3. **Aggregation under homogeneous signals.** Having established conditions for good aggregation under signal diversity, we now explore what happens without signal diversity. Our general message is that aggregation is worse.

To gain an intuition for this, note that it is essential to the argument described in the previous subsection that different agents have different signal precisions. Recall the complete graph case. From the perspective of an agent $(i, t+1)$, the fact that type A and type B neighbors place different weights on the social signal r_t allows the agent to avoid a collinearity problem and separate θ_t from a confound. In that example, with the same types of private signals the type A and B agents would use the same weights, and our agent would face a collinearity problem.

We begin by studying graphs having a symmetric structure and show that learning outcomes are necessarily bounded very far from good aggregation. We then turn to arbitrary large graphs and prove a lower-bound on aggregation error that implies the homogeneous-signals regime has, quite generally, worse outcomes for some agents than those achieved by everyone in our good-aggregation result.

4.3.1. *Aggregation in graphs with symmetric neighbors.*

[24]Time-zero agents have only their own signals to use, so they play simple linear strategies, and subsequent agents inductively set their weights based on the prior ones, as described in Section 3.1.

Definition 3. A network G has *symmetric neighbors* if $N_j = N_{j'}$ for any i and any $j, j' \in N_i$.

In the undirected case, the graphs with symmetric neighbors are the complete network and complete bipartite networks.[25] For directed graphs, the condition allows a larger variety of networks.

Consider a sequence $(G_n)_{n=1}^{\infty}$ of strongly connected graphs with symmetric neighbors. Assume that all signal qualities are the same, equal to σ^2, and that $m = 1$.

Proposition 2. *Under the assumptions in the previous paragraph, each G_n has a unique equilibrium. There exists $\varepsilon > 0$ such that the ε-aggregation benchmark is not achieved by any agent i at this equilibrium for any n.*

All agents have non-vanishing aggregation errors at the unique equilibrium. So all agents learn poorly compared to the diverse signals case. The proof of this proposition, and the proofs of all subsequent results, appear in Appendix F.

This failure of good aggregation is not due simply to a lack of sufficient information in the environment: On the complete graph with exchangeable (i.e., non-diverse) signals, a social planner who exogenously set weights for all agents could achieve ε-aggregation for any $\varepsilon > 0$ when n is large. See Appendix H for a formal statement, proof and numerical results.[26] In this sense, the social learning externalities are quite severe: a fairly small change in weights for each individual could yield a very large benefit in a world of homogeneous signal types.

We now give intuition for Proposition 2. In a graph with symmetric neighbors, in the unique equilibrium, the actions of any agent's neighbors are exchangeable.[27] So actions must be unweighted averages of observations. This prevents the sort of inference of θ_t that occurred with diverse signals. This is easiest to see on the complete graph, where *all* observations are exchangeable. So, in any equilibrium, each agent's action at time $t+1$ is equal to a weighted average of his own signal and $\frac{1}{|N_i|}\sum_{j \in N_i} a_{j,t}$:

$$a_{i,t+1} = \widehat{w}_i^s s_{i,t+1} + (1 - \widehat{w}_i^s) \frac{1}{|N_i|} \sum_{j \in N_i} a_{j,t}. \tag{4.2}$$

By iteratively using this equation, we can see that actions must place substantial weight on the average of signals from, e.g., two periods ago. Although the effect of *signal errors* $\eta_{i,t}$ vanishes as n grows large, the correlated error from *past changes in the state* never "washes out" of estimates, and this is what prevents vanishing aggregation errors.

[25]These are both special cases of our stochastic block model from Section 4.2.
[26]We thank Alireza Tahbaz-Salehi for suggesting this analysis.
[27]The proof of the proposition establishes uniqueness by showing that Φ is a contraction in a suitable sense.

We can also explicitly characterize the limit action variances and covariances. Consider again the complete graph and the (unique) symmetric equilibrium. Let V^∞ denote the limit, as n grows large, of the variance of any agent's error $(a_{i,t} - \theta_t)$. Let Cov^∞ denote the limit covariance of any two agent's errors. By direct computations, these can be seen to be related by the following equations, which have a unique solution:

$$V^\infty = \frac{1}{\sigma^{-2} + (\rho^2 Cov^\infty + 1)^{-1}}, \qquad Cov^\infty = \frac{(\rho^2 Cov^\infty + 1)^{-1}}{[\sigma^{-2} + (\rho^2 Cov^\infty + 1)^{-1}]^2}. \tag{4.3}$$

This variance and covariance describe behavior not only in the complete graph, but in any graph with symmetric neighbors where degrees tend uniformly to ∞. In such graphs, too, the variances of all agents converge to V^∞ and the covariances of all pairs of agents converge to Cov^∞, as $n \to \infty$. This implies that, in large graphs, the equilibrium action distributions are close to symmetric. Indeed, it can be deduced that these actions are equal to an appropriately discounted sum of past $\theta_{t-\ell}$, up to error terms (arising from $\eta_{i,t-\ell}$) that vanish asymptotically.

As a consequence of Theorem 1 and Proposition 2, we can give an example where making one agent's private information less precise helps all agents.

Corollary 2. *There exists a network G and an agent $i \in G$ such that increasing σ_i^2 gives a Pareto improvement at the unique equilibrium.*

To prove the corollary, we consider the complete graph with homogeneous signals and n large. By Proposition 2, all agents have non-vanishing aggregation errors. If we instead give agent 1 a very uninformative signal, all players can anti-imitate agent 1 and achieve vanishing aggregation errors. When the signals at the initial configuration are sufficiently imprecise, this gives a Pareto improvement.

4.3.2. *Aggregation in arbitrary graphs.* Section 4.3.1 showed aggregation errors are non-vanishing when signal endowments and neighborhoods are symmetric. A natural question is whether asymmetry in network positions can substitute for asymmetry in signal endowments. In Section 4.2 the key point was that different neighbors' actions were informative about different linear combinations of θ_t and θ_{t-1}, and this permitted filtering. Perhaps different network positions can achieve the same effect?

We thus move to *arbitrary* networks and show a weaker but much more general result. Consider any sequence of equilibria on any networks with symmetric signal endowments. No equilibrium achieves C/n-aggregation for almost all agents, no matter what C is. In particular, this implies that the rate of learning is slower than at the good-learning equilibrium with diversity of signal endowments from Theorem 1.

Theorem 2. *Let $C > 0$. Let $(G_n)_{n=1}^{\infty}$ be an arbitrary sequence of networks and suppose all private signals have variance σ^2. Then, in any sequence of equilibria, $\widehat{\kappa}_i^2 > C/n$ for a positive fraction of agents i.*

In addition to considering arbitrary networks, we allow the memory m to be arbitrary (yet finite). Because the assumptions are much weaker, we obtain a weaker conclusion than in Proposition 2. . While Proposition 2 shows that aggregation errors are non-vanishing, the theorem shows that aggregation errors cannot vanish quickly but does not rule out aggregation errors vanishing more slowly.

The basic intuition is that to avoid putting substantial weight on θ_{t-2}, an agent at time t must anti-imitate some neighbors. If all or almost all neighbors achieve C/n-aggregation for some C, there is not much diversity among neighbors. So more and more anti-imitation is needed as n grows large in the sense that the total positive weight and total negative weight on neighbors both grow large. But then the contribution to the agent's variance from neighbors' private signal errors cannot vanish quickly.

We can combine Theorems 1 and 2 to compare the value of signal diversity and network diversity. With diversity of signal endowments, there exists $C > 0$ such that asymptotically almost surely there is a good-learning equilibrium achieiving the C/n-aggregation benchmark for all agents under the stochastic block model. With exchangeable signals, there is no such sequence under *any* sequence of networks.

5. The importance of understanding correlations

In the positive result on achieving the C/n-aggregation benchmark (Theorem 1), a key aspect of the argument involved agents filtering out confounding information from their neighbors' estimates—i.e., responding in a sophisticated way to the correlation structure of those estimates. In this section, we demonstrate that this sort of behavior is essential for nearly perfect aggregation, and that more naively imitative heuristics yield outcomes far from the benchmark. Empirical studies have found evidence (depending on the setting and the subjects) consistent with both equilibrium behavior and naive inference in the presence of correlated observations (e.g., Eyster, Rabin, and Weizsacker, 2015; Dasaratha and He, 2019; Enke and Zimmermann, 2019).

We begin with a canonical model of agents who do not account for correlations among their neighbors' estimates conditional on the state, and show by example that naive agents achieve much worse learning than Bayesian agents, and thus have non-vanishing aggregation errors. We then formalize the idea that accounting for correlations in neighbors' actions is crucial to reaching the benchmark. This is done by demonstrating a general lack of asymptotic learning by agents who use imitative strategies, rather than filtering in a sophisticated way. Finally, we show that even in fixed, finite networks, any positive weights chosen by optimizing agents will be Pareto-dominated.

5.1. **Naive agents.** In this part we introduce agents who misunderstand the distribution of the signals they are facing and who therefore do not update as Bayesians with a correct understanding of their environment. We consider a particular form of misspecification that simplifies solving for equilibria analytically:[28]

Definition 4. We call an agent *naive* if she believes that all neighbors choose actions equal to their private signals and maximizes her expected utility given these incorrect beliefs.

Equivalently, a naive agent believes her neighbors all have empty neighborhoods. This is the analogue, in our model, of "best-response trailing naive inference" (Eyster and Rabin, 2010). So naive agents understand that their neighbors' actions from the previous period are estimates of θ_{t-1}. But they think each such estimate is independent given the state, and that the precision of the estimate is equal to the signal precision of the corresponding agent. They then play their expectation of the state given this misspecified theory of others' play.

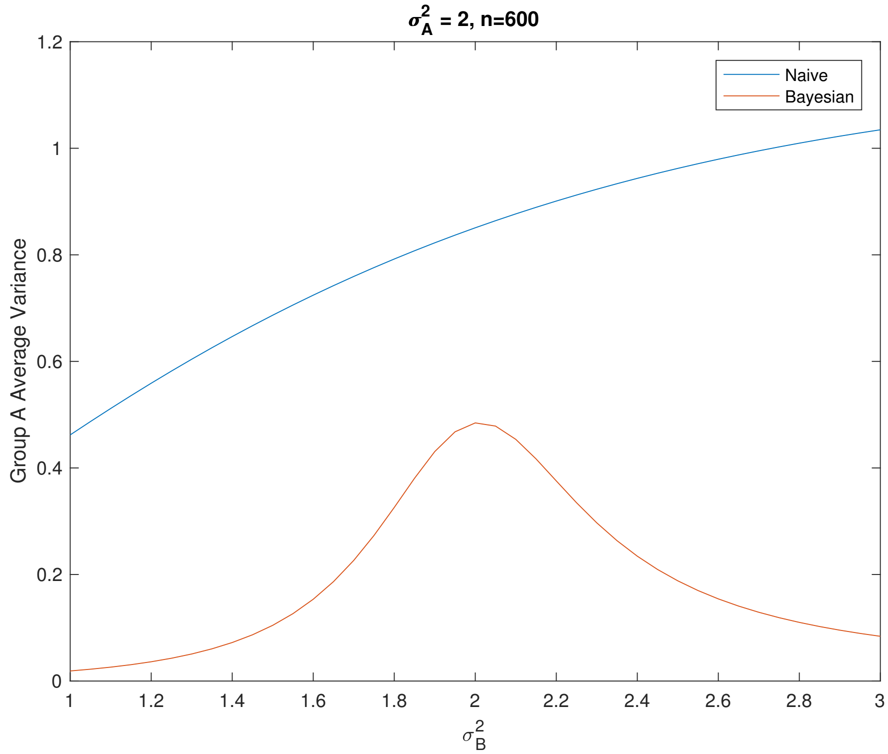
In Figure 5.1, we compare Bayesian and naive learning outcomes. We consider a complete network with 600 agents and $\alpha = 0.9$. Half of agents have signal variance $\sigma_A^2 = 2$, while we vary the signal variance σ_B^2 of the remaining agents. The figure shows the average social signal variance for the group of agents with private signal variance $\sigma_A^2 = 2$. We observe that naive agents learn substantially worse than rational agents, whether signals are diverse or not. Formal analysis and formulas for variances under naive learning can be found in Appendix G.

5.2. **More general learning rules: Understanding correlation is essential for good aggregation.** We now show more generally that a sophisticated response to correlation is needed to achieve vanishing aggregation errors on any sequence of growing networks. To this end, we make the following definition:

Definition 5. The *steady state* associated with weights $\boldsymbol{W}$ and $\boldsymbol{w}^s$ is the (unique) covariance matrix $\boldsymbol{V}^*$ such that if actions have a variance-covariance matrix given by $\boldsymbol{V}_t = \boldsymbol{V}^*$ and next-period actions are set using weights $(\boldsymbol{W}, \boldsymbol{w}^s)$, then $\boldsymbol{V}_{t+1} = \boldsymbol{V}^*$ as well.

[28]There are a number of possible variants of our behavioral assumption, and it is straightforward to numerically study alternative specifications of behavior in our model (Alatas et al., 2016 consider one such variant).

FIGURE 5.1. Bayesian and Naive Learning



In this definition of steady state, instead of best-responding to others' actual distributions of play, agents use exogenous weights $\boldsymbol{W}$ in all periods.

By a straightforward application of the contraction mapping theorem, if agents use any non-negative weights under which covariances remain bounded at all times, there is a unique steady state.

Consider a sequence of networks $(G_n)_{n=1}^{\infty}$ with n agents in G_n.

Proposition 3. *Fix any sequence of steady states under non-negative weights on G_n. Suppose that all private signal variances are bounded below by $\underline{\sigma}^2 > 0$ and that all agents place weight at most $\overline{w} < 1$ on their private signals. Then there is an $\varepsilon > 0$ such that, for all n, the ε-aggregation benchmark is not achieved by any agent i at steady state.*

The essential idea is that at time $t+1$ observed time-t actions all put weight on actions from period $t-1$, which causes θ_{t-1} to have a (positive weight) contribution to all observed actions. Agents do not know θ_{t-1} and, with positive weights, cannot take any linear combination that would recover it. Even with a very large number of observations, this confound prevents agents from learning yesterday's state precisely.

To see why the weights on private signals must be bounded away from one, note that an individual agent could learn well without adjusting for correlations by observing many autarkic agents who simply report their private signals. But in this case, all of these autarkic agents would have non-vanishing aggregation errors. Without the bound on private signal weights, *some* agent must still fail to achieve the ϵ-aggregation benchmark for small enough ϵ.

On undirected networks, the proposition implies that aggregation errors do not vanish under naive inference or under various other specifications of non-Bayesian inference. Moreover, the same argument shows that in any sequence of *Bayesian* equilibria on undirected networks where all agents use positive weights, no agent can learn well.

5.3. **Without anti-imitation, outcomes are Pareto-inefficient.** The previous section argued that anti-imitation is critical to achieving vanishing aggregation errors. We now show that even in small networks, where that benchmark is not relevant, any equilibrium without anti-imitation is Pareto-inefficient relative to another steady state. This result complements our asymptotic analysis by showing a different sense (relevant for small networks) in which anti-imitation is necessary to make the best use of information.

Proposition 4. *Suppose the network G is strongly connected and some agent has more than one neighbor. Given any naive equilibrium or any Bayesian equilibrium where all weights are positive, the action variances at that equilibrium are Pareto-dominated by action variances at another steady state.*

The basic argument behind Proposition 4 is that if agents place marginally more weight on their private signals, this introduces more independent information that eventually benefits everyone. In the proof in Appendix F, we state and prove a more general result with weaker hypotheses on behavior.

In a review of sequential learning experiments, Weizsäcker (2010) finds that subjects weight their private signals more heavily than is optimal (given the empirical behavior of others they observe). Proposition 4 implies that in our environment with optimizing agents, it is actually welfare-improving for individuals to "overweight" their own information relative to best-response behavior.

Discussion of conditions in the proposition. We next briefly discuss the sufficient conditions in the proposition statement. First, it is clear that some condition on neighborhoods is needed: If every agent has exactly one neighbor and updates rationally or naively, there are no externalities and the equilibrium weights are Pareto optimal.[29] Second, the condition on equilibrium weights says that no agent anti-imitates any of her neighbors. This assumption makes the analysis tractable, but we believe the basic force also works in finite networks with some anti-imitation.

Proof sketch. The idea of the proof of the rational case is to begin at the steady state and then marginally shift the rational agent's weights toward her private signal. By the envelope theorem, this means agents' actions are less correlated but not significantly worse in the next period. We show that if all agents continue using these new weights, the decreased correlation eventually benefits everyone. In the last step, we use the absence of anti-imitation, which implies that the updating function associated with agents using fixed (as opposed to best-response) weights is monotonic in terms of the variances of guesses. To first order, some covariances decrease while others do not change after one period under the new weights. Monotonicity of the updating function and strong connectedness imply that eventually all agents' variances decrease.

The proof in the naive case is simpler. Here a naive agent is overconfident about the quality of her social information, so she would benefit from shifting some weight from her social information to her signal. This deviation also reduces her correlation with other agents, so it is Pareto-improving.

An illustration. An example illustrates the phenomenon:

Example 3. Consider $n = 100$ agents in an undirected circle—i.e., each agent observes the agent to her left and the agent to her right. Let $\sigma_i^2 = \sigma^2$ be equal for all agents and $\rho = .9$. The equilibrium strategies place weight $\widehat{w}^s$ on private signals and weight $\frac{1}{2}(1 - \widehat{w}^s)$ on each observed action.

When $\sigma^2 = 10$, the equilibrium weight is $\widehat{w}^s = 0.192$ while the welfare-maximizing symmetric weights have $w^s = 0.234$. That is, weighting private signals substantially more is Pareto improving. When $\sigma^2 = 1$, the equilibrium weight is $\widehat{w}^s = 0.570$ while the welfare maximizing symmetric weights have $w^s = 0.586$. The inefficiency persists, but the equilibrium strategy is now closer to the optimal strategy.

6. Related literature

The question of whether decentralized communication can facilitate efficient adaptation to a changing world is a fundamental one in economic theory, related to questions raised

[29]In fact, the result of Proposition 4 (with the same proof) applies to a larger class of networks: rather than strongly connectedness it is sufficient that, starting at each agent, there are two paths of some length k to a rational agent and another distinct agent.

by Hayek (1945)[30] and of primary interest in some applied problems, e.g., in real business cycle models with consumers and firms learning about evolving states. Nevertheless, there is relatively little modeling of Bayesian learning of dynamic states in the large literature on social learning and information aggregation in networks. We now put our contribution in the context of the most closely related papers on social learning in networks.[31]

Play in the stationary linear equilibria of our model closely resembles behavior in the DeGroot (1974) model, where agents update by linearly aggregating network neighbors' past estimates, with constant weights on neighbors over time. DeMarzo, Vayanos, and Zweibel (2003), in an environment with an unchanging state, derive DeGroot learning as a boundedly-rational heuristic. Each agent has one initial signal about the state, and this determines her $t = 0$ estimate, which is shared with neighbors. Assuming all randomness is Gaussian, the Bayesian rule for forming estimates at $t = 1$ is linear with certain weights. DeMarzo, Vayanos, and Zweibel (2003) made the behavioral assumption that in subsequent periods, agents treat the informational environment as being identical to that of the first period—even though past learning has, in fact, induced redundancies and correlations. Molavi, Tahbaz-Salehi, and Jadbabaie (2018) have offered new bounded-rationality foundations for the DeGroot rule. We give an alternative, Bayesian microfoundation for the same sort of rule by studying a different environment. Our foundation relies on the fact that the stationary environment admits a stationary equilibrium in which fixed updating rules are best responses.[32] In the introduction, we have discussed the contrast between outcomes in our model and in fixed-state benchmarks.

Several recent papers in computer science and engineering study dynamic environments similar to ours. Shahrampour, Rakhlin, and Jadbabaie (2013) study an exogenous-weights version, interpreted as a set of Kalman filters under the control of a planner; they focus on computing or bounding various measures of welfare in terms of network invariants and the persistence of the state process (ρ). Frongillo, Schoenebeck, and Tamuz (2011) study (in our notation) a θ_t that follows a random walk ($\rho = 1$). They examine agents who learn using fixed, exogenous weights on arbitrary networks. They characterize the steady-state

[30] "If we can agree that the economic problem of society is mainly one of rapid adaptation to changes in the particular circumstances of time and place... there still remains the problem of communicating to [each individual] such further information as he needs." Hayek's main concern was aggregation of information through markets, but the same questions apply more generally.

[31] For more complete surveys of different parts of this literature, see, among others, Acemoglu and Ozdaglar (2011), Golub and Sadler (2016), and Mossel and Tamuz (2017). See Moscarini, Ottaviani, and Smith (1998) for an early model in a binary-action environment, where it is shown that a changing state can break information cascades.

[32] Indeed, agents behaving according to the DeGroot heuristic even when it is not appropriate might have to do with their experiences in stationary environments where it is closer to optimal.

distribution of behavior with arbitrary (non-equilibrium) fixed weights. They also give a formula for equilibrium weights on a complete network (where everybody observes everybody else) and show these weights are inefficient. Our Proposition 4 on Pareto-inefficiency on an arbitrary network documents a related inefficiency. Our main question—the quality of equilibrium learning in large networks—is a topic not considered in these papers.

Alatas et al. (2016) perform an empirical exercise using a learning model related to ours. There, agents are not Bayesian, ignoring the correlation between social observations, similarly to our naive models. The paper's focus is estimating parameters of social learning rules using data from Indonesian villages. (The state variables there are the wealths of villagers.) We theoretically analyze both rational and naive behavior and show that the degree of rationality can be pivotal for the outcomes of such processes. Our model also provides foundations for structural estimation of Bayesian learning behavior as well as testing of the Bayesian model against behavioral alternatives such as that of Alatas et al. (2016); we discuss this below in Section 7.2.

Some recent learning models—e.g., Sethi and Yildiz (2012) and Harel, Mossel, Strack, and Tamuz (2019)—consider other obstacles to learning in environments with Gaussian signals and a fixed state. In Sethi and Yildiz (2012), learning outcomes depend on whether individuals' (heterogeneous) priors are independent or correlated. Harel, Mossel, Strack, and Tamuz (2019), among others, study the rate of social learning in a Gaussian setting. Bad learning corresponds to this rate being low. In our setting, we can focus on steady-state errors, which provide an alternative measure of the quality of learning.

Finally, a robust aspect of rational learning in sequential models is the phenomenon of anti-imitation, as discussed, e.g., by Eyster and Rabin (2014). They give general conditions for fully Bayesian agents to anti-imitate in the sequential model. We find that anti-imitation is also an important feature in our dynamic model, and in our context is crucial for good learning. Despite this similarity, there is an important contrast between our environment and standard sequential models. In those models, while rational agents *do* prefer to anti-imitate, in many cases individuals, and society as a whole, could obtain good outcomes using heuristics without any anti-imitation: for instance, by combining the information that can be inferred from a single neighbor with one's own private signal. Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) and Lobel and Sadler (2015) show that such a heuristic leads to asymptotic learning in a sequential model. Our dynamic learning environment is different, as shown in Proposition 3: to have any hope of approaching good aggregation benchmarks, agents must respond in a sophisticated way, with anti-imitation, to their neighbors' (correlated) estimates.

7. Discussion and extensions

7.1. **Aggregation and its absence without asymptotics: Numerical results.** The message of Section 4 is that signal diversity enables good aggregation, and signal homogeneity obstructs it. The theoretical results in that section, however, were asymptotic, and the good-aggregation result used some assumptions on the distribution of graphs. In this section we show that the substantive message applies to realistic networks with moderate degrees. We do this by computing equilibria for actual social networks from the data in Banerjee, Chandrasekhar, Duflo, and Jackson (2013). This data set contains the social networks of villages in rural India.[33] There are 43 networks in the data, with an average network size of 212 nodes (standard deviation = 53.5), and an average degree of 19 (standard deviation = 7.5).

Our simulation exercises measure the benefits of heterogeneity for equilibrium aggregation, holding constant the total amount of information that reaches the community via private signals. For each network, we calculate the equilibrium with $\rho = 0.9$ for two types of environments. The first is the homogeneous case, with all signal variances set to 2. The second is a heterogeneous case, where half of villagers have a signal variance less than 2 and half of villagers have a signal variance greater than 2. We choose these signal variances so that the average precision in each village is $\frac{1}{2}$, as in the homogeneous case. This signal assignment holds fixed the average utility when all villagers are autarkic, or equivalently holds fixed the average utility when all villagers know the state θ_{t-1} in the previous period exactly. At the same time, it varies the level of heterogeneity in signal endowments. Villagers are randomly assigned to better or worse private signals, and the simulation results do not depend substantially on the realized random assignment. Our outcomes will be the average social signal error variance in each village and the average social signal error variance across all villages.

It is useful to begin by looking at the equilibrium average aggregation errors, i.e., social signal variances, in the case of homogeneous signals. This is the horizontal coordinate in Figure 7.1(a); each village is a data point, and the points have a standard deviation of 0.013. In this case, differences in learning outcomes are due only to differences in the network structure, and we will call this number the *network-driven variation.* Now we introduce some private signal diversity diversity. In our first exercise, we change the variance of the worse private signal from 2 (homogeneous signals) to 3 (heterogeneous signals), and

[33]We take the networks that were used in the estimation in Banerjee, Chandrasekhar, Duflo, and Jackson (2013). As in their work, we take every reported relationship to be reciprocal for the purposes of sharing information. This makes the graphs undirected.

adjust the other variance as discussed above to hold fixed the total amount of information coming into the network. The vertical coordinate in Figure 7.1(b) depicts the equilibrium aggregation error in each village. The average of this number across all villages falls to 0.470, compared to from 0.555 (in the homogeneous case). Therefore, adding heterogeneity by increasing the private signal variance for half of the agents by 50% changes social signal error variance by 6.5 times the network-driven variation. Learning is much better with some private signal heterogeneity than in villages with very favorable networks (i.e., those that achieve the best aggregation under homogeneous signals).

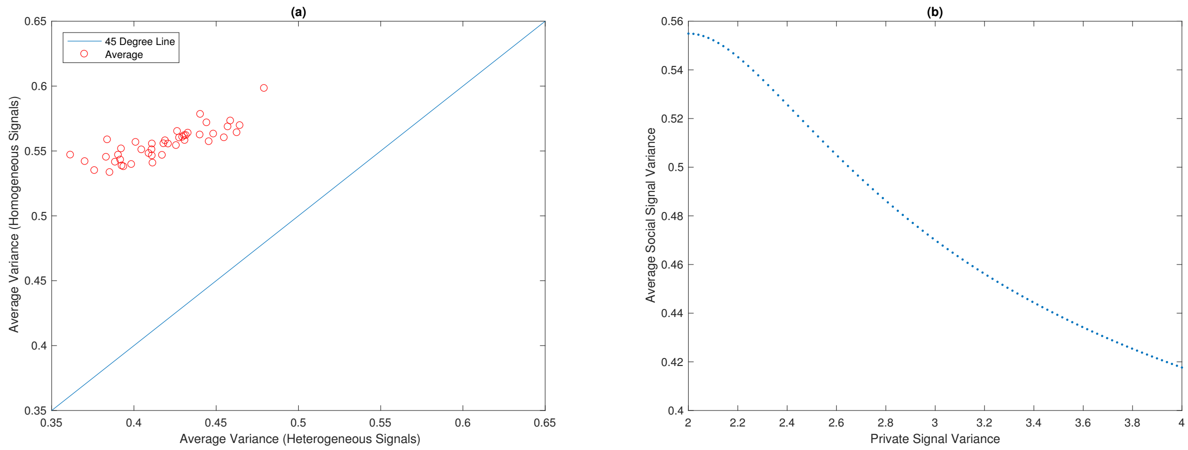
In Figure 7.1(b), rather than working with the particular choice of 3 for the variance of the private signal, we look across all choices of this variance between 2 and 4 and plot the average equilibrium social signal variance across all villages.

Figure 7.1(b) also sheds light on the value of a small amount of heterogeneity. The results in Section 4 can be summarized as saying that, to achieve the aggregation benchmark of essentially knowing the previous period's state, there need to be at least two different private signal variances in the network. Formally, this is a knife-edge result: As long as private signal variances differ at all, then as $n \to \infty$, aggregation errors vanish; with exactly homogeneous signal endowments, aggregation errors are much higher. The figure shows that the transition from the first regime to the second is actually gradual. In particular, a very small amount of heterogeneity provides little benefit in finite networks, as there is not enough diversity of signal endowments for villagers to anti-imitate. However, a 50% change in the variance of one of the signals (equivalently, a 22% change in its standard deviation) makes the community much better able to use the same total amount of information.

We also conduct simulations on Erdos-Renyi random graphs in Appendix E. These graphs have expected degrees between 25 and 100, which are much larger than the observed degrees in the village networks but comparable to other empirical social networks. We use the same variances as in Figure 7.1(a). We find that compared to the hetereogeneous case, average social signal variances in the homogeneous case are more than twice as large with expected degree 50 and more than 3 times as large with expected degree 100.

7.2. **Identification and testable implications.** One of the main advantages of the parametrization we have studied is that standard methods can easily be applied to estimate the model and test hypotheses within it. The key feature making the model econometrically well-behaved is that, in the solutions we focus on, agents' actions are linear functions of the random variables they observe. Moreover, the evolution of the state and arrival of information creates exogenous variation. We briefly sketch how these features can be used for estimation and testing.

FIGURE 7.1. Social Signal Variance In Indian Villages



(a) The average social signal variance of agents in each village, in the homogeneous and heterogeneous cases. In the homogeneous case all agents have private signal variance 2. In the heterogeneous case, half of agents have private signal variance $\frac{3}{2}$ and half of agents have private signal variance 3. (b) The average social signal variance for all agents as we vary the worse private signal variance from 2 to 4 and hold fixed the average precision of private signals.

Assume the following. The analyst obtains noisy measurements $\overline{a}_{i,t} = a_{i,t} + \xi_{i,t}$ of agent's actions (where $\xi_{i,t}$ are i.i.d., mean-zero error terms). He knows the parameter ρ governing the stochastic process, but may not know the network structure or the qualities of private signals $(\sigma_i)_{i=1}^n$. Suppose also that the analyst observes the state θ_t ex post (perhaps with a long delay).[34]

Now, consider *any* steady state in which agents put constant weights W_{ij} on their neighbors and w_i^s on their private signals over time. We will discuss the case of $m = 1$ to save on notation, though all the statements here generalize readily to arbitrary m.

We first consider how to estimate the weights agents are using, and to back out the structural parameters of our model when it applies. The strategy does not rely on uniqueness of equilibrium. We can identify the weights agents are using through standard vector autoregression methods. In steady state,

$$\overline{a}_{i,t} = \sum_j W_{ij} \rho \overline{a}_{j,t-1} + w_i^s \theta_t + \zeta_{i,t}, \tag{7.1}$$

[34]We can instead assume that the analyst observes (a proxy for) the private signal $s_{i,t}$ of agent i; we mention how below.

where $\zeta_{i,t} = w_i^s \eta_{i,t} - \sum_j W_{ij} \rho \xi_{j,t-1} + \xi_{i,t}$ are error terms i.i.d. across time. The first term of this expression for $\zeta_{i,t}$ is the error of the signal that agent i receives at time t. The summation combines the measurement errors from the observations $\overline{a}_{j,t-1}$ from the previous period.[35] Thus, we can obtain consistent estimators $\widetilde{W}_{ij}$ and $\widetilde{w}_i^s$ for W_{ij} and w_i^s, respectively.

We now turn to the case in which agents are using *equilibrium* weights. First, and most simply, our estimates of agents' equilibrium weights allow us to recover the network structure. If the weight $\widehat{W}_{ij}$ is non-zero for any i and j, then agent i observes agent j. Generically the converse is true: if i observes j then the weight $\widehat{W}_{ij}$ is non-zero. Thus, network links can generically be identified by testing whether the recovered social weights are nonzero. For such tests (and more generally) the standard errors in the estimators can be obtained by standard techniques.[36]

Now we examine the more interesting question of how structural parameters can be identified assuming an equilibrium is played, and also how to test the assumption of equilibrium.

The first step is to compute the empirical covariances of action errors from observed data; we call these $\widetilde{V}_{ij}$. Under the assumption of equilibrium, we now show how to determine the signal variances using the fact that equilibrium is characterized by $\Phi(\widehat{\boldsymbol{V}}) = \widehat{\boldsymbol{V}}$ and recalling the explicit formula (3.3) for Φ. In view of this formula, the signal variances σ_i^2 are uniquely determined by the other variables:

$$\widehat{V}_{ii} = \sum_j \sum_k \widehat{W}_{ij} \widehat{W}_{ik} (\rho^2 \widehat{V}_{jk} + 1) + (\widehat{w}_i^s)^2 \sigma_i^2. \tag{7.2}$$

Replacing the model parameters other than σ_i^2 by their empirical analogues, we obtain a consistent estimate $\widetilde{\sigma}_i^2$ of σ_i. This estimate could be directly useful—for example, to an analyst who wants to choose an "expert" from the network and ask about her private signals directly.

Note that our basic VAR for recovering the weights relies only on constant linear strategies and does not assume that agents are playing any particular strategy within this class. Thus, if agents are using some other behavioral rule (e.g., optimizing in a misspecified model) we can replace (7.2) by a suitable analogue that reflects the bounded rationality in agents' inference. If such a steady state exists, and using the results in this section, one can create an econometric test that is suitable for testing how agents are behaving. For

[35] This system defines a VAR(1) process (or generally VAR(m) for memory length m).

[36] Methods involving regularization may be practically useful in identifying links in the network. Manresa (2013) proposes a regularization (LASSO) technique for identifying such links (peer effects). In a dynamic setting such as ours, with serial correlation, the techniques required will generally be more complicated.

instance, we can test the hypothesis that they are Bayesian against the naive alternative of our Section 5.1.

7.3. **Time-varying networks.** Our model works with a network G that is fixed over time, corresponding to the idea that observation opportunities reflect geographic or organizational structure. It is possible, however, to extend the analysis to networks that vary over time, with each generation drawing its own links (according to some distribution). Many of our techniques would have analogues in this setting. Under suitable ergodicity assumptions, one could define a steady-state analogue of the stationary equilibrium, though the weights agents use would evolve with the network. We also conjecture that, for example, the good-aggregation results of Section 4.2 would continue to hold with little modification; indeed, a version of such a result is sketched in the next subsection.

Relative to this richer model, we view the model we have focused on as a convenient starting point; it is also most directly comparable to classic bechmarks such as the DeGroot model. Extensions such as the one we have sketched here seem like a natural direction for further research.

7.4. **Multidimensional states and informational specialization.** So far, we have been working with a one-dimensional state and one-dimensional signals, which varied only in their precisions. Our message about the value of diversity is, however, better interpreted in a mathematically equivalent multidimensional model.

Consider Bayesian agents who learn and communicate about two independent dimensions simultaneously (each one working as in our model). If all agents have equally precise signals about both dimensions, then society may not learn well about either of them. In contrast, if half the agents have superior signals about one dimension and inferior signals about the other (and the other half has the reverse), then society can learn well about both dimensions. Thus, the designer has a strong preference for an organization with informational specialization where some, but not all, agents are expert in a particular dimension.[37]

Of course, there are many familiar reasons for specialization, in information or any other activity. For instance, it may be that more total information can be collected in this case, or that incentives are easier to provide. Crucially, specialization is valuable in our setting for a reason distinct from all these: it helps agents with their inference problems.

[37]This raises important questions about what information agents would acquire, and whom they would choose to observe, which are the focus of a growing literature. For recent papers on this in the context of networks, see Sethi and Yildiz (2016) and Myatt and Wallace (2017), among others.

7.5. **General distributions.** The example of the previous subsection involved trivially extending our model to several independent dimensions. We now briefly discuss a more substantive extension, which applies to more realistic signal structures.

Our analysis of stationary linear learning rules relied crucially on the assumptions that the innovations ν_t and signal errors $\eta_{i,t}$ are Gaussian random variables. However, we believe the basic logic of our result about good aggregation with signal diversity (Theorem 1) does not depend on this particular distributional assumption, or the exact functional form of the AR(1) process.

Suppose we have

$$\theta_t = \mathbb{T}(\theta_{t-1}, \nu_t) \quad \text{and} \quad s_{i,t} = \mathbb{S}(\theta_t, \eta_t)$$

and consider more general distributions of innovations ν_t and signal errors η_t. For simplicity, consider the complete graph and $m = 1$.[38] Because θ_{t-1} is still a sufficient statistic for the past, an agent's action in period t will still be a function of her subjective distribution over θ_{t-1} and her private signal. An agent with type τ (which is observable) who believes θ_{t-1} is distributed according to $\mathcal{D}$ takes an action equal to $f(\tau, \mathcal{D}, s_{i,t})$. Here, τ could reflect the distribution of agent i's signal, but also perhaps her preferences. We no longer assume that an agent's action is her posterior mean of the random variable: it might be some other statistic, and might be multi-dimensional. Similarly, information need not be one-dimensional, or characterized only by its precision.

This framework gives an abstract identification condition: agents can learn well if, for any feasible distribution $\mathcal{D}$ over θ_{t-1}, the state θ_t can be inferred from the observed distributions of actions, i.e., distribution of $(\tau, f(\tau, \mathcal{D}, s_{i,t}))$, which each agent would essentially know given enough observations.[39]

Now consider a time-t agent i. Suppose now that any possible distribution that time-$(t-1)$ agents might have over θ_{t-2} can be fully described by a finite tuple of parameters $d \in \mathbb{R}^p$ (e.g., a finite number of moments). For each type τ of $t-1$ agent that i observes, the distribution of $f(\tau, d, s_{i,t})$ gives an agent a different measurement of d, which is a summary of beliefs about θ_{t-2}, and θ_{t-1}. Assuming there is not too much "collinearity," these measurements of the finitely many parameters of interest should provide linearly

[38] These states and signals may now be multidimensional.

[39] The easiest way to ensure this is to consider a complete network and a large number of agents. However, notice that even if the neighborhood of an agent changes from period to period, if some of the individuals in that neighborhood are randomly sampled each period, this can provide information about the empirical distribution of actions of various types in the same way. That, in turn, can facilitate the identification of recent states, as we are about to explain.

independent information about θ_{t-1}. Thus, as long as the set of signal types τ is sufficiently rich, we would expect the identification condition to hold.

The simplest example of this is one in which the state is still AR(1) Gaussian, but now d-dimensional. Private signals, now also multidimensional, are arbitrary linear functions of θ_t. If these linear functions are generic and there are sufficiently many distinct signal types, then observing actions will allow an observer to back out both θ_{t-2}, and θ_{t-1}. The essential observation about what is needed for good learning by Bayesians is that there are enough linearly independent action rules to identify the underlying d dimensions of fundamental uncertainty. In the Gaussian case such linear independence is guaranteed by having at least $d+1$ generic signal types.

References

Acemoglu, D., M. Dahleh, I. Lobel, and A. Ozdaglar (2011): "Bayesian Learning in Social Networks," *The Review of Economic Studies*, 78, 1201–1236.

Acemoglu, D. and A. Ozdaglar (2011): "Opinion dynamics and learning in social networks," *Dynamic Games and Applications*, 1, 3–49.

Alatas, V., A. Banerjee, A. G. Chandrasekhar, R. Hanna, and B. A. Olken (2016): "Network structure and the aggregation of information: Theory and evidence from Indonesia," *The American Economic Review*, 106, 1663–1704.

Babus, A. and P. Kondor (2018): "Trading and Information Diffusion in Over-the-Counter Markets," *Econometrica*, 86, 1727–1769.

Bala, V. and S. Goyal (1998): "Learning from neighbours," *The Review of Economic Studies*, 65, 595–621.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006): "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, 74, 1403–1417.

Banerjee, A., A. G. Chandrasekhar, E. Duflo, and M. O. Jackson (2013): "The diffusion of microfinance," *Science*, 341, 1236498.

Banerjee, A. and D. Fudenberg (2004): "Word-of-mouth learning," *Games and economic behavior*, 46, 1–22.

Dasaratha, K. and K. He (2019): "An experiment on network density and sequential learning," *arXiv preprint arXiv:1909.02220.*

DeGroot, M. H. (1974): "Reaching a consensus," *Journal of the American Statistical Association*, 69, 118–121.

DeMarzo, P., D. Vayanos, and J. Zweibel (2003): "Persuasion bias, social influence, and unidimensional opinions," *The Quarterly Journal of Economics*, 118, 909–968.

Enke, B. and F. Zimmermann (2019): "Correlation neglect in belief formation," *The Review of Economic Studies*, 86, 313–332.

Eyster, E. and M. Rabin (2010): "Naive herding in rich-information settings," *American Economic Journal: Microeconomics*, 2, 221–243.

——— (2014): "Extensive Imitation is Irrational and Harmful," *Quarterly Journal of Economics*, 129, 1861–1898.

Eyster, E., M. Rabin, and G. Weizsacker (2015): "An Experiment on Social Mislearning," *Working Paper*.

Frongillo, R., G. Schoenebeck, and O. Tamuz (2011): "Social Learning in a Changing World," *Internet and Network Economics*, 146–157.

Golub, B. and M. Jackson (2010): "Naive learning in social networks and the wisdom of crowds," *American Economic Journal: Microeconomics*, 2, 112–149.

Golub, B. and E. Sadler (2016): "Learning in Social Networks," in *The Oxford Handbook of the Economics of Networks*, ed. by Y. Bramoullé, A. Galeotti, B. Rogers, and B. Rogers, Oxford University Press, chap. 19, 504–542.

Griliches, Z. (1957): "Hybrid corn: An exploration in the economics of technological change," *Econometrica, Journal of the Econometric Society*, 501–522.

Harel, M., E. Mossel, P. Strack, and O. Tamuz (2019): "Groupthink and the Failure of Information Aggregation in Large Groups," *arXiv preprint arXiv:1412.7172*.

Hayek, F. A. (1945): "The use of knowledge in society," *The American Economic Review*, 35, 519–530.

Holland, P. W., K. B. Laskey, and S. Leinhardt (1983): "Stochastic blockmodels: First steps," *Social Networks*, 5, 109–137.

Jadbabaie, A., P. Molavi, A. Sandroni, and A. Tahbaz-Salehi (2012): "Non-Bayesian social learning," *Games and Economic Behavior*, 76, 210–225.

Kay, S. M. (1993): *Fundamentals of Statistical Signal Processing*, Prentice Hall PTR.

Lambert, N. S., M. Ostrovsky, and M. Panov (2018): "Strategic trading in informationally complex environments," *Econometrica*, 86, 1119–1157.

Lobel, I. and E. Sadler (2015): "Information diffusion in networks through social learning," *Theoretical Economics*, 10, 807–851.

Manresa, E. (2013): "Estimating the structure of social interactions using panel data," *Unpublished Manuscript. CEMFI, Madrid*.

Molavi, P., A. Tahbaz-Salehi, and A. Jadbabaie (2018): "A Theory of Non-Bayesian Social Learning," *Econometrica*, 86, 445–490.

Moscarini, G., M. Ottaviani, and L. Smith (1998): "Social Learning in a Changing World," *Economic Theory*, 11, 657–665.

Mossel, E., M. Mueller-Frank, A. Sly, and O. Tamuz (2018): "Social learning equilibria," ArXiv:1207.5895.

Mossel, E. and O. Tamuz (2017): "Opinion exchange dynamics," *Probability Surveys*, 14, 155–204.

Myatt, D. and C. Wallace (2017): "Information Acquisition and Use by Networked Players," University of Warwick. Department of Economics. CRETA Discussion Paper Series (32), available at http://wrap.warwick.ac.uk/90449/.

Pinelis, I. (2018): "Inverse of matrix with blocks of ones," MathOverflow, URL:https://mathoverflow.net/q/296933 (version: 2018-04-04).

Sethi, R. and M. Yildiz (2012): "Public disagreement," *American Economic Journal: Microeconomics*, 4, 57–95.

——— (2016): "Communication with unknown perspectives," *Econometrica*, 84, 2029–2069.

Shahrampour, S., S. Rakhlin, and A. Jadbabaie (2013): "Online learning of dynamic parameters in social networks," *Advances in Neural Information Processing Systems*.

Vives, X. (1993): "How fast do rational agents learn?" *The Review of Economic Studies*, 60, 329–347.

Weizsäcker, G. (2010): "Do we follow others when we should? A simple test of rational expectations," *The American Economic Review*, 100, 2340–2360.

Wolitzky, A. (2018): "Learning from Others' Outcomes," *American Economic Review*, 108, 2763–2801.

Appendix A. Details of definitions

A.1. **Exogenous random variables.** Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\nu_t, \eta_{i,t})_{t\in\mathbb{Z}, i\in N}$ be normal, mutually independent random variables, with ν_t having variance 1 and $\eta_{i,t}$ having variance σ_i^2. Also take a stochastic process $(\theta_t)_{t\in\mathbb{Z}}$, such that for each $t \in \mathbb{Z}$, we have (for $0 < |\rho| \leq 1$)

$$\theta_t = \rho\theta_{t-1} + \nu_t.$$

Such a stochastic process exists by standard constructions of the AR(1) process or, in the case of $\rho = 1$, of the Gaussian random walk on a doubly infinite time domain. Define $s_{i,t} = \theta_t + \eta_{i,t}$.

A.2. **Formal definition of game and stationary linear equilibria.**

Players and strategies. The set of players (or agents) is $\mathcal{A} = \{(i, t) : i \in N, t \in \mathbb{Z}\}$. The set of (pure) *responses* of an agent (i, t) is defined to be the set of all Borel-measurable functions $\xi_{(i,t)} : \mathbb{R}\times(\mathbb{R}^{|N(i)|})^m \to \mathbb{R}$, mapping her own signal and her neighborhood's actions, $(s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m)$, to a real-valued action $a_{i,t}$. We call the set of these functions $\widetilde{\Xi}_{(i,t)}$. Let $\widetilde{\Xi} = \prod_{(i,t)\in\mathcal{A}} \widetilde{\Xi}_{(i,t)}$ be the set of response profiles. We now define the set of *(unambiguous) strategy profiles*, $\Xi \subset \widetilde{\Xi}$. We say that a response profile $\xi \in \widetilde{\Xi}$ is a strategy profile if the following two conditions hold

1. There is a tuple of real-valued random variables $(a_{i,t})_{i\in N,t\in\mathbb{Z}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $(i, t) \in \mathcal{A}$, we have
$$a_{i,t} = \xi_{(i,t)}\left(s_{i,t}, (\boldsymbol{a}_{N_i,t-\ell})_{\ell=1}^m\right).$$
2. Any two tuples of real-valued random variables $(a_{i,t})_{i\in N,t\in\mathbb{Z}}$ satisfying Condition 1 are equal almost surely.

That is, a response profile is a strategy profile if there is an essentially unique specification of behavior that is consistent with the responses: i.e., if the responses uniquely determine the behavior of the population, and hence payoffs.[40] Note that if $\xi \in \Xi$, then it can be checked that $\widetilde{\xi} = (\xi'_{(i,t)}, \xi_{-(i,t)}) \in \Xi$ whenever $\xi'_{(i,t)} \in \widetilde{\Xi}_{(i,t)}$. Thus, if we start with a strategy profile and consider agent (i,t)'s deviations, they are unrestricted: she may consider any response.

Payoffs. The payoff of an agent (i,t) under any strategy profile $\xi \in \Xi$ is

$$u_{i,t}(\xi) = -\mathbb{E}\left[(a_{i,t} - \theta_t)^2\right] \in [-\infty, 0],$$

where the actions $a_{i,t}$ are taken according to $\xi_{(i,t)}$ and the expectation is taken in the probability space we have described. This expectation is well-defined because inside the expectation there is a nonnegative, measurable random variable, for which an expectation is always defined, though it may be infinite.

Equilibria. A (Nash) *equilibrium* is defined to be a strategy profile $\xi \in \Xi$ such that, for each $(i,t) \in \mathcal{A}$ and each $\widetilde{\xi} \in \Xi$ such that $\widetilde{\xi} = (\xi'_{(i,t)}, \xi_{-(i,t)})$ for some $\xi'_{(i,t)} \in \Xi_{(i,t)}$, we have

$$u_{i,t}(\widetilde{\xi}) \leq u_{i,t}(\xi).$$

For $p \in \mathbb{Z}$, we define the shift operator $\mathfrak{T}_p$ to translate variables to time indices shifted p steps forward. This definition may be applied, for example, to Ξ.[41] A strategy profile $\xi \in \Xi$ is *stationary* if, for all $p \in \mathbb{Z}$, we have $\mathfrak{T}_p \xi = \xi$.

We say $\xi \in \Xi$ is a *linear* strategy profile if each ξ_i is a linear function. Our analysis focuses on *stationary, linear equilibria.*

Appendix B. Existence of equilibrium: Proof of Proposition 1

Recall from Section 3.1 the map Φ, which gives the next-period covariance matrix $\Phi(\boldsymbol{V}_t)$ for any $\boldsymbol{V}_t$. The expression given there for this map ensures that its entries are continuous functions of the entries of $\boldsymbol{V}_t$. Our strategy is to show that this function maps a convex, compact set, $\mathcal{K}$, to itself, which, by Brouwer's fixed-point theorem, ensures that Φ has a fixed point $\widehat{\boldsymbol{V}}$. We will then argue that this fixed point corresponds to a stationary linear equilibrium.

We begin by defining the compact set $\mathcal{K}$. Because memory is arbitrary, entries of $\boldsymbol{V}_t$ are covariances between pairs of neighbor actions from any periods available in memory. Let k, l be two indices of such actions, corresponding to actions taken at nodes i and j

[40] Condition 1 is necessary to rule out response profiles such as the one given by $\xi_{i,t}(s_{i,t}, a_{i,t-1}) = |a_{i,t-1}| + 1$. This profile, despite consisting of well-behaved functions, does not correspond to any specification of behavior for the whole population (because time extends infinitely backward). Condition 2 is necessary to rule out response profiles such has the one given by $\xi_{i,t}(s_{i,t}, a_{i,t-1}) = a_{i,t-1}$, which have many satisfying action paths, leaving payoffs undetermined.

[41] I.e., $\sigma' = \mathfrak{T}_p \sigma$ is defined by $\sigma_{(i,t)} = \sigma_{(i,t-p)}$.

respectively (at potentially different times), and let $\overline{\sigma}_i^2 = \max\{\sigma_i^2, \rho^{m-1}\sigma_i^2 + \frac{1-\rho^{m-1}}{1-\rho}\}$. Now let $\mathcal{K} \subset \mathcal{V}$ be the subset of symmetric positive semi-definite matrices $\boldsymbol{V}_t$ such that, for any such k, l,

$$V_{kk,t} \in \left[\min\left\{\frac{1}{1+\sigma_i^{-2}}, \frac{\rho^{m-1}}{1+\sigma_i^{-2}} + \frac{1-\rho^{m-1}}{1-\rho}\right\}, \max\left\{\sigma_i^2, \rho^{m-1}\sigma_i^2 + \frac{1-\rho^{m-1}}{1-\rho}\right\}\right]$$
$$V_{kl,t} \in [-\overline{\sigma}_i\overline{\sigma}_j, \overline{\sigma}_i\overline{\sigma}_j].$$

This set is closed and convex, and we claim that $\Phi(\mathcal{K}) \subset \mathcal{K}$.

To show this claim, we will first find upper and lower bounds on the variance of any neighbor's action (at any period in memory). For the upper bound, note that a Bayesian agent will not choose an action with a larger variance than her signal, which has variance σ_i^2. For a lower bound, note that if she knew the previous period's state and her own signal, then the variance of her action would be $\frac{1}{1+\sigma_i^{-2}}$. Thus an agent observing only noisy estimates of θ_t and her own signal can do no better.

By the same reasoning applied to the node-i agent from m periods ago, the error variance of $\rho^m a_{i,t-m} - \theta_t$ is at most $\rho^m\sigma_i^2 + \frac{1-\rho^m}{1-\rho}$ and at least $\frac{\rho^m}{1+\sigma_i^{-2}} + \frac{1-\rho^m}{1-\rho}$. This establishes bounds on $V_{kk,t}$ for observations k from either the most recent or the oldest available period. The corresponding bounds from the periods between $t-m+1$ and t are always weaker than at least one of the two bounds we have described, so we need only take minima and maxima over two terms.

This established the claimed bound on the variances. The bounds on covariances follow from Cauchy-Schwartz.

We have now established that there is a variance-covariance matrix $\widehat{\boldsymbol{V}}$ such that $\Phi(\widehat{\boldsymbol{V}}) = \widehat{\boldsymbol{V}}$. By definition of Φ, this means there exists some weight profile $(\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{w}}^{\boldsymbol{s}})$ such that, when applied to prior actions that have variance-covariance matrix $\widehat{\boldsymbol{V}}$, produce variance-covariance matrix $\widehat{\boldsymbol{V}}$. However, it still remains to show that this is the variance-covariance matrix reached when agents have been using the weights $(\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{w}}^{\boldsymbol{s}})$ forever.

To show this, first observe that if agents have been using the weights $(\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{w}}^{\boldsymbol{s}})$ forever, the variance-covariance matrix $\boldsymbol{V}_t$ in any period is uniquely determined and does not depend on t; call this $\check{\boldsymbol{V}}$.[42] This is because actions can be expressed as linear combinations of private signals with coefficients depending only on the weights. Second, it follows from our construction above of the matrix $\widehat{\boldsymbol{V}}$ and the weights $(\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{w}}^{\boldsymbol{s}})$ that there is a distribution

[42]The variance-covariance matrices are well-defined because the (W, w^s) weights yield unambiguous strategy profiles in the sense of Appendix A.

of actions where the variance-covariance matrix is $\widehat{\boldsymbol{V}}$ in every period and agents are using weights $(\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{w}}^{\boldsymbol{s}})$ in every period. Combining the two statements shows that in fact $\check{\boldsymbol{V}} = \widehat{\boldsymbol{V}}$, and this completes the proof. Note that this argument also establishes that the response profile we have constructed is a strategy profile: under the responses used, we can write formally the dependence of actions on all prior signals, and verify using the observations on decay of dependence across time that the formula is summable and hence defines unique actions.

Appendix C. Proof of Theorem 1

C.1. **Notation and key notions.** Let $\mathbb{S}$ be the (by assumption finite) set of all possible signal variances, and let $\overline{\sigma}^2$ be the largest of them. The proof will focus on the covariances of errors in social signals. Suppose that all agents have at least one neighbor. Take two arbitrary agents i and j. Recall that both $r_{i,t}$ and $r_{j,t}$ have mean θ_{t-1}, because each is an unbiased estimate[43] of θ_{t-1}; we will thus focus on the errors $r_{i,t} - \theta_{t-1}$. Let A_t denote the variance-covariance matrix $(\mathrm{Cov}(r_{i,t} - \theta_{t-1}, r_{j,t} - \theta_{t-1}))_{i,j}$ and let $\mathcal{W}$ be the set of such covariance matrices. For all i, j note that $\mathrm{Cov}(r_{i,t} - \theta_{t-1}, r_{j,t} - \theta_{t-1}) \in [-\overline{\sigma}^2, \overline{\sigma}^2]$ using the Cauchy-Schwarz inequality and the fact that $\mathrm{Var}(r_{i,t} - \theta_{t-1}) \in [0, \overline{\sigma}^2]$ for all i. This fact about variances says that no social signal is worse than putting all weight on an agent who follows only her private signal. Thus the best-response map Φ is well-defined and induces a map $\widetilde{\Phi}$ on $\mathcal{W}$.

Next, for any $\psi, \zeta > 0$ we will define the subset $\mathcal{W}_{\psi,\zeta} \subset \mathcal{W}$ to be the set of covariance matrices in $\mathcal{W}$ such that both of the following hold:

1. for any pair of distinct agents[44] $i \in G_n^k$ and $j \in G_n^{k'}$,
$$\mathrm{Cov}(r_{i,t} - \theta_{t-1}, r_{j,t} - \theta_{t-1}) = \psi_{kk'} + \zeta_{ij}$$
where (i) $\psi_{kk'}$ depends only on the network types of the two agents (k and k', which may be the same); (ii) $|\psi_{kk'}| < \psi$; and (iii) $|\zeta_{ij}| < \zeta$;
2. for any single agent $i \in G_n^k$,
$$\mathrm{Var}(r_{i,t} - \theta_{t-1}) = \psi_k + \zeta_{ii}$$
where (i) ψ_k only depends on the network type of the agent; (ii) $|\psi_k| < \psi$, and (iii) $|\zeta_{ii}| < \zeta$.

[43]This is because it is a linear combination, with coefficients summing to 1, of unbiased estimates of θ_{t-1}.
[44]Throughout this proof, we abuse terminology by referring to agents and nodes interchangeably when the relevant t is clear or specified nearby.

This is the space of covariance matrices such that each covariance is split into two parts. Considering (1) first, $\psi_{kk'}$ is an effect that depends only on i's and j's network types, while ζ_{ij} adjusts for the individual-level heterogeneity arising from different link realizations. The description of the decomposition in (2) is analogous.

C.2. **Proof strategy.**

C.2.1. *A set $\mathcal{W}_{\overline{\psi},\overline{\zeta}}$ of outcomes with good learning.* Our goal is to show that as n grows large, there is an equilibrium in which $\mathrm{Var}(r_{i,t} - \theta_{t-1})$ becomes very small, which then implies that the agents asymptotically learn. To this end we define a set of covariances with this property as well as some other useful properties. We will take $\overline{\psi}$ and $\overline{\zeta}$ to be arbitrarily small numbers and show that for large enough n, with high probability (which we abbreviate "asymptotically almost surely" or "a.a.s.") there is an equilibrium with a social error covariance matrix A_t in the set $\mathcal{W}_{\overline{\psi},\overline{\zeta}}$. That will imply that, in this equilibrium, $\mathrm{Var}(r_{i,t} - \theta_{t-1})$ becomes arbitrarily small as we take the constants $\overline{\psi}$ and $\overline{\zeta}$ to be small. In our constructions, the ζ_{ij} (resp., ζ_i) terms will be set to much smaller values than the $\psi_{kk'}$ (resp., ψ_k) terms, because group-level covariances are more predictable and less sensitive to idiosyncratic realizations than individual-level covariances.

C.2.2. *Approach to showing that $\mathcal{W}_{\overline{\psi},\overline{\zeta}}$ contains an equilibrium.* To show that there is (a.a.s.) an equilibrium outcome with a social error covariance matrix A_t in the set $\mathcal{W}_{\overline{\psi},\overline{\zeta}}$, the plan is to construct a set so that (a.a.s.) $\overline{\mathcal{W}} \subset \mathcal{W}_{\overline{\psi},\overline{\zeta}}$ and $\widetilde{\Phi}(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$. This set will contain an equilibrium by the Brouwer fixed point theorem, and therefore so will $\mathcal{W}_{\overline{\psi},\overline{\zeta}}$.

To construct the set $\overline{\mathcal{W}}$, we will fix a positive constant β (to be determined later), and define

$$\overline{\mathcal{W}} = \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}} \cup \widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}).$$

We will then prove that, for large enough n, (i) $\widetilde{\Phi}(\overline{\mathcal{W}}) \subseteq \overline{\mathcal{W}}$ and (ii) for another suitable positive constant λ,

$$\overline{\mathcal{W}} \subset \mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}.$$

This will allow us to establish that (a.a.s.) $\overline{\mathcal{W}} \subset \mathcal{W}_{\overline{\psi},\overline{\zeta}}$ and $\widetilde{\Phi}(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$, with $\overline{\psi}$ and $\overline{\zeta}$ being arbitrarily small numbers.

The following two lemmas will allow us to deduce (immediately after stating them) properties (i) and (ii) of $\overline{\mathcal{W}}$.

Lemma 1. *There is a function $\underline{\lambda}(\beta) \geq 1$ such that the following holds. For all large enough β and all $\lambda \geq \underline{\lambda}(\beta)$, for n sufficiently large we have $\widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}$ with probability at least $1 - \frac{1}{n}$.*

Lemma 2. *For all large enough β, for n sufficiently large, $\widetilde{\Phi}^2(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$, with probability at least $1-\frac{1}{n}$.*[45]

Putting these lemmas together, a.a.s. we have,

$$\widetilde{\Phi}^2(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}} \quad \text{and} \quad \widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}.$$

From this it follows that $\overline{\mathcal{W}} = \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}} \cup \widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}})$ is mapped to a subset of itself by $\widetilde{\Phi}$, and contained in $\mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}$, as claimed.

C.2.3. *Proving the lemmas by analyzing how $\widetilde{\Phi}$ and $\widetilde{\Phi}^2$ act on sets $\mathcal{W}_{\psi,\zeta}$.* The lemmas are about how $\widetilde{\Phi}$ and $\widetilde{\Phi}^2$ act on the covariance matrix A_t, assuming it is in a certain set $\mathcal{W}_{\psi,\zeta}$, to yield new covariance matrices. Thus, we will prove these lemmas by studying two periods of updating. The analysis will come in five steps.

Step 1: No-large-deviations (NLD) networks and the high-probability event. Step 1 concerns the "with high probability" part of the lemmas. In the entire argument, we condition on the event of a *no-large-deviations (NLD)* network realization, which says that certain realized statistics in the network (e.g., number of paths between two nodes) are close to their expectations. The expectations in question depend only on agents' types. Therefore, on the NLD realization, the realized statistics do not vary much based on which exact agents we focus on, but rather depend only on their types. Step 1 defines the NLD event E formally and shows that it has high probability. We use the structure of the NLD event throughout our subsequent steps, as we mention below.

Step 2: Weights in one step of updating are well-behaved. We are interested in $\widetilde{\Phi}$ and $\widetilde{\Phi}^2$, which describe how the covariance matrix A_t of social signal errors changes under updating. How this works is determined by the "basic" updating map Φ, and so we begin by studying the weights involved in it and then make deductions about the implications for the evolution of the variance-covariance matrixA_t.

The present step establishes that in one step of updating, the weight $W_{ij,t+1}$ that agent $(i, t+1)$ places on the action of another agent j in period t, does not depend too much on the identities of i and j. It only depends on their (network and signal) types. This is established by using our explicit formula for weights in terms of covariances. We rely on (i) the fact that covariances are assumed to start out in a suitable $\mathcal{W}_{\psi,\zeta}$, and (ii) our conditioning on the NLD event E. The NLD event is designed so that the network quantities that go into determining the weights depend only on the types of i and j (because the NLD event forbids

[45]The notation $\widetilde{\Phi}^2$ means the operator $\widetilde{\Phi}$ applied twice.

too much variation within type). The restriction to $A_t \in \mathcal{W}_{\psi,\zeta}$ ensures that covariances in the initial period t do not vary too much with type, either.

Step 3: Lemma 1: $\widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}$. Once we have analyzed one step of updating, it is natural to consider the implications for the covariance matrix. Because we now have a bound on how much weights can vary after one step of updating, we can compute bounds on covariances. We show that if covariances A_t are in $\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$, then after one step, covariances are in $\mathcal{W}_{\frac{\beta}{n},\frac{\lambda}{n}}$. Note that the introduction of another parameter λ on the right-hand side implies that this step might worsen our control on covariances somewhat, but in a bounded way. This establishes Lemma 1.

Step 4: Weights in two steps of updating are well-behaved. The fourth step establishes that the statement made in Step 2 remains true when we replace $t+1$ by $t+2$. By the same sort of reasoning as in Step 2, an additional period of updating cannot create too much further idiosyncratic variation in weights. Proving this requires analyzing the covariance matrices of various social signals (i.e., the A_{t+1} that the updating induces), which is why we needed to do Step 3 first.

Step 5: Lemma 2: $\widetilde{\Phi}^2(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}) \subset \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$. Now we use our understanding of weights from the previous steps, along with additional structure, to show the key remaining fact. What we have established so far about weights allows us to control the weight that a given agent's estimate at time $t+2$ places on the social signal of another agent at time t. This is Step 5(a). In the second part, Step 5(b), we use that to control the covariances in A_{t+2}. It is important in this part of the proof that different agents have very similar "second-order neighborhoods": the paths of length 2 beginning from an agent are very similar, in terms of their counts and what types of agents they go through. We use our control of second-order neighborhoods, as well as the assumptions on variation across entries of A_t to bound this variation well enough to conclude that $A_{t+2} \in \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$.

C.3. Carrying out the steps.

C.3.1. *Step 1.* Here we formally define the NLD event, which we call E. It is given by $E = \cap_{i=1}^{5} E_i$, where the events E_i will be defined next.

(E_1) Let $X_{i,\tau k}^{(1)}$ be the number of agents having signal type τ and network type k who are observed by i. The event E_1 is that this quantity is close to its expected value in the following sense, simultaneously for all possible values of the subscript:

$$(1-\zeta^2)\mathbb{E}[X_{i,\tau k}^{(1)}] \leq X_{i,\tau k}^{(1)} \leq (1+\zeta^2)\mathbb{E}[X_{i,\tau k}^{(1)}].$$

(E_2) Let $X_{ii',\tau k}^{(2)}$ be the number of agents having signal type τ and network type k who are observed by *both* i and i'. The event E_2 is that this quantity is close to its expected value in the following sense, simultaneously for all possible values of the subscript:

$$(1-\zeta^2)\mathbb{E}[X^{(2)}_{ii',\tau k}] \leq X^{(2)}_{ii',\tau k} \leq (1+\zeta^2)\mathbb{E}[X^{(2)}_{ii',\tau k}].$$

(E_3) Let $X^{(3)}_{i,\tau k,j}$ be the number of agents having signal type τ and network type k who are observed by agent i and who observe agent j. The event E_3 is that this quantity is close to its expected value in the following sense, simultaneously for all possible values of the subscript:

$$(1-\zeta^2)\mathbb{E}[X^{(3)}_{i,\tau k,j}] \leq X^{(3)}_{i,\tau k,j} \leq (1+\zeta^2)\mathbb{E}[X^{(3)}_{i,\tau k,j}].$$

(E_4) Let $X^{(4)}_{ii',\tau k,j}$ be the number of agents having signal type τ and network type k who are observed by both agent i and i' and who observe j. The event E_4 is that this quantity is close to its expected value in the following sense, simultaneously for all possible values of the subscript:

$$(1-\zeta^2)\mathbb{E}[X^{(4)}_{ii',\tau k',j}] \leq X^{(4)}_{ii',\tau k',j} \leq (1+\zeta^2)\mathbb{E}[X^{(4)}_{ii',\tau k',j}].$$

(E_5) Let $X^{(5)}_{i,\tau k,jj'}$ be the number of agents of signal type τ and network type k who are observed by agent i and who observe both j and j'. The event E_5 is that this quantity is close to its expected value in the following sense, simultaneously for all possible values of the subscript:

$$(1-\zeta^2)\mathbb{E}[X^{(5)}_{i,\tau k,jj'}] \leq X^{(5)}_{i,\tau k,jj'} \leq (1+\zeta^2)\mathbb{E}[X^{(5)}_{i,\tau k,jj'}].$$

We claim that the probability of the complement of the event E vanishes exponentially. We can check this by showing that the probability of each of the E_i vanishes exponentially. For E_1, for example, the bounds will hold unless at least one agent has degree outside the specified range. The probability of this is bounded above by the sum of the probabilities of each individual agent having degree outside the specified range. By Chebyshev's inequality, the probability a given agent has degree outside this range vanishes exponentially. Because there are n agents in G_n, this sum vanishes exponentially as well. The other cases are similar.

For the rest of the proof, we condition on the event E.

C.3.2. *Step 2.* As a shorthand, let $\psi = \beta/n$ for a sufficiently large constant β, and let $\zeta = 1/n$.

Lemma 3. *Suppose that in period t the matrix $A = A_t$ of covariances of social signals satisfies $A \in \mathcal{W}_{\psi,\zeta}$ and all agents are optimizing in period $t+1$. Then there is a γ so that for all n sufficiently large,*

$$\frac{W_{ij,t+1}}{W_{i'j',t+1}} \in \left[1-\frac{\gamma}{n}, 1+\frac{\gamma}{n}\right].$$

whenever i and i' have the same network and signal types and j and j' have the same network and signal types.

To prove this lemma, we will use our weights formula:

$$W_{i,t+1} = \frac{\mathbf{1}^T \boldsymbol{C}_{i,t}^{-1}}{\mathbf{1}^T \boldsymbol{C}_{i,t}^{-1} \mathbf{1}}.$$

This says that in period $t+1$, agent i's weight on agent j is proportional to the sum of the entries of column j of $\boldsymbol{C}_{i,t}^{-1}$. We want to show that the change in weights is small as the covariances of observed social signals vary slightly. To do so we will use the Taylor expansion of $f(A) = \boldsymbol{C}_{i,t}^{-1}$ around the covariance matrix $A(0)$ at which all $\psi_{kk'} = 0$, $\psi_k = 0$ and $\zeta_{ij} = 0$.

We begin with the first partial derivative of f at $A(0)$ in an arbitrary direction. Let $A(x)$ be any perturbation of A_0 in one parameter, i.e., $A(x) = A(0) + xM$ for some constant matrix M with entries in $[-1, 1]$. Let $\boldsymbol{C}_i(x)$ be the matrix of covariances of the actions observed by i given that the covariances of agents' social signals were $A(x)$. There exists a constant γ_1 depending only on the possible signal types such that each entry of $\boldsymbol{C}_i(x) - \boldsymbol{C}_i(x')$ has absolute value at most $\gamma_1(x - x')$ whenever both x and x' are small.

We will now show that the column sums of $\boldsymbol{C}_i(x)^{-1}$ are close to the column sums of $\boldsymbol{C}(0)_i^{-1}$. To do so, we will evaluate the formula

$$\frac{\partial f(A(x))}{\partial x} = \frac{\partial \boldsymbol{C}_i(x)^{-1}}{\partial x} = \boldsymbol{C}_i(x)^{-1} \frac{\partial \boldsymbol{C}_i(x)}{\partial x} \boldsymbol{C}_i(x)^{-1} \tag{C.1}$$

at zero. If we can bound each column sum of this expression (evaluated at zero) by a constant (depending only on the signal types and the number of network types K), then the first derivative of f will also be bounded by a constant.

Recall that $\mathbb{S}$ is the set of signal types and let $S = |\mathbb{S}|$; index the signal types by numbers ranging from 1 to S. To bound the column sums of $\boldsymbol{C}_i(0)^{-1}$, suppose that the agent observes r_i agents from each signal type $1 \leq i \leq S$. Reordering so that all agents of each signal type are grouped together, we can write

$$\boldsymbol{C}_i(0) = \begin{pmatrix} a_{11}\mathbf{1}_{r_1\times r_1} + b_1 I_{r_1} & a_{12}\mathbf{1}_{r_1\times r_2} & & a_{S1}\mathbf{1}_{r_1\times r_S} \\ a_{12}\mathbf{1}_{r_2\times r_1} & a_{22}\mathbf{1}_{r_2\times r_2} + b_2 I_{r_2} & & \vdots \\ & & \ddots & \\ a_{1S}\mathbf{1}_{r_S\times r_1} & \cdots & & a_{SS}\mathbf{1}_{r_s\times r_s} + b_s I_{r_s} \end{pmatrix}$$

Therefore, $\boldsymbol{C}_i(0)$ can be written as a block matrix with blocks $a_{ij}1_{r_i\times r_j} + b_i\delta_{ij}I_{r_i}$ where $1 \leq i, j \leq S$ and $\delta_{ij} = 1$ for $i = j$ and 0 otherwise.

We now have the following important approximation of the inverse of this matrix.[46]

Lemma 4 (Pinelis (2018)). *Let C be a matrix consisting of $S \times S$ blocks, with its (i,j) block given by*

$$a_{ij}1_{r_i \times r_j} + b_i \delta_{ij} I_{r_i}$$

and let $A = a_{ij}1_{r_i \times r_j}$ be an invertible matrix. As $n \to \infty$, then the (i,i) block of C^{-1} is equal to

$$\frac{1}{b_i} I_{r_i} - \frac{1}{b_i r_i} 1_{r_i \times r_i} + O(1/n^2)$$

while the off-diagonal blocks are $O(1/n^2)$.

Proof. First note that the ij-block of C^{-1} has the form

$$c_{ij}1_{r_i \times r_j} + d_i \delta_{ij} I_{r_i}$$

for some real c_{ij} and d_i.

Therefore, CC^{-1} can be written in matrix form as

$$\begin{aligned} &\textstyle\sum_k (a_{ik}1_{r_i \times r_k} + b_i \delta_{ik} I_{r_i})(c_{kj}1_{r_k \times r_j} + d_k \delta_{kj} I_{r_k}) = \\ &(a_{ij}d_j + \textstyle\sum_k (a_{ik} r_k + \delta_{ik} b_k) c_{kj})\, 1_{r_i \times r_j} + b_i d_i \delta_{ij} I_{r_i}. \end{aligned} \tag{C.2}$$

Note that the last summand is the identity matrix.

Let D_d denote the diagonal matrix with d_i in the (i,i) diagonal entry, let $D_{1/b}$ denote the diagonal matrix with $1/b_i$ in the (i,i) diagonal entry, etc. Breaking up the previous display (C.2) into its diagonal and off-diagonal parts, we can write

$$AD_d + (AD_r + D_b)C = 0 \text{ and } D_d = D_{1/b}.$$

Hence,

$$\begin{aligned} C &= -(AD_r + D_b)^{-1} AD_d \\ &= -(I_q + D_r^{-1} A^{-1} D_b)^{-1} (AD_r)^{-1} AD_{1/b} \\ &= -(I_q + D_r^{-1} A^{-1} D_b)^{-1} D_{1/(br)} \\ &= -D_{1/(br)} + O(1/n^2) \end{aligned}$$

[46] We are very grateful to Iosif Pinelis for suggesting this argument.

where $br := (b_1 r_1, \dots, b_q r_q)$. Therefore as $n \to \infty$ the off-diagonal blocks will be $O(1/n^2)$ while the diagonal blocks are

$$\frac{1}{b_i} I_{r_i} - \frac{1}{b_i r_i} 1_{r_i \times r_i} + O(1/n^2)$$

as desired. □

Using Lemma 4 we can analyze the column sums of[47]

$$\boldsymbol{C}_i(0)^{-1} M \boldsymbol{C}_i(0)^{-1}.$$

In more detail, we use the formula of the lemma to estimate both copies of $\boldsymbol{C}_i(0)^{-1}$, and then expand this to write an expression for any column sum of $\boldsymbol{C}_i(0)^{-1} M \boldsymbol{C}_i(0)^{-1}$. It follows straightforwardly from this calculation that all these column sums are $O(1/n)$ whenever all entries of M are in $[-1, 1]$.

We can bound the higher-order terms in the Taylor expansion by the same technique: by differentiating equation C.1 repeatedly in x, we obtain an expression for the k^{th} derivative in terms of $\boldsymbol{C}_i(0)^{-1}$ and M:

$$f^{(k)}(0) = k! \boldsymbol{C}_i(0)^{-1} M \boldsymbol{C}_i(0)^{-1} M \boldsymbol{C}_i(0)^{-1} \cdot \ldots \cdot M \boldsymbol{C}_i(0)^{-1},$$

where M appears k times in the product. By the same argument as above, we can show that the column sums of $\frac{f^{(k)}(0)}{k!}$ are bounded by a constant independent of n. The Taylor expansion is

$$f(A) = \sum_k \frac{f^{(k)}(0)}{k!} x^k.$$

Since we take $A \in \mathcal{W}_{\psi,\zeta}$, we can assume that x is $O(1/n)$. Because the column sums of each summand are bounded by a constant times x^k, the column sums of $f(A)$ are bounded by a constant.

Finally, because the variation in the column sums is $O(1/n)$ and the weights are proportional to the column sums, each weight varies by at most a multiplicative factor of γ_1/n for some γ_1. We find that the first part of the lemma, which bounded the ratios between weights $W_{ij,t+1}/W_{i'j',t+1}$, holds.

C.3.3. *Step 3.* We complete the proof of Lemma 1, which states that the covariance matrix of $r_{i,t+1}$ is in $\mathcal{W}_{\psi,\zeta'}$. Recall that $\zeta' = \lambda/n$ for some constant n, so we are showing that if the covariance matrix of the $r_{i,t}$ is in a neighborhood $\mathcal{W}_{\psi,\zeta}$, then the covariance matrix in the

[47] Recall we wrote $A(x) = A(0) + xM$, and in (C.1) we expressed the derivative of f in x in terms of the matrix we exhibit here.

next period is in a somewhat larger neighborhood $\mathcal{W}_{\psi,\zeta'}$. The remainder of the argument then follows by the same arguments as in the proof of the first part of the lemma: we now bound the change in time-$(t+2)$ weights as we vary the covariances of time-$(t+1)$ social signals within this neighborhood.

Recall that we decomposed each covariance $\mathrm{Cov}(r_{i,t}-\theta_{t-1}, r_{j,t}-\theta_{t-1}) = \psi_{kk'} + \zeta_{ij}$ into a term $\psi_{kk'}$ depending only on the types of the two agents and a term ζ_{ij}, and similarly for variances. To show the covariance matrix is contained in $\mathcal{W}_{\psi,\zeta'}$, we bound each of these terms suitably.

We begin with ζ_{ij} (and ζ_i). We can write

$$r_{i,t+1} = \sum_j \frac{W_{ij,t+1}}{1-w^s_{i,t+1}} a_{i,t} = \sum_j \frac{W_{ij,t+1}}{1-w^s_{i,t+1}} (w^s_{j,t} s_{j,t} + (1-w^s_{j,t}) r_{j,t}).$$

By the first part of the lemma, the ratio between any two weights (both of the form $W_{ij,t+1}$, $w^s_{i,t+1}$, or $w^s_{j,t}$) corresponding to pairs of agents of the same types is in $[1-\gamma_1/n, 1+\gamma_1/n]$ for a constant γ_1. We can use this to bound the variation in covariances of $r_{i,t+1}$ within types by ζ': we take the covariance of $r_{i,t+1}$ and $r_{j,t+1}$ using the expansion above and then bound the resulting summation by bounding all coefficients.

Next we bound $\psi_{kk'}$ (and ψ_k). It is sufficient to show that $\mathrm{Var}(r_{i,t+1}-\theta_t)$ is at most ψ. To do so, we will give an estimator of θ_t with variance less than β/n, and this will imply $\mathrm{Var}(r_{i,t+1}-\theta_t) < \beta/n = \psi$ (recall $r_{i,t+1}$ is the estimate of θ_t given agent i's social observations in period $t+1$). Since this bounds all the variance terms by ψ, the covariance terms will also be bounded by ψ in absolute value.

Fix an agent i of network type k and consider some network type k' such that $p_{kk'} > 0$. Then there exists two signal types, which we call A and B, such that i observes $\Omega(n)$ agents of each of these signal types in G^k_n.[48] The basic idea will be that we can approximate θ_t well by taking a linear combination of the average of observed agents of network type k and signal type A and the average of observed agents of network type k and signal type B.

In more detail: Let $N_{i,A}$ be the set of agents of type A in network type k observed by i and $N_{i,B}$ be the set of agents of type B in network type k observed by i. Then fixing some agent j_0 of network type k,

$$\frac{1}{|N_{i,A}|} \sum_{j \in N_{i,A}} a_{j,t-1} = \frac{\sigma_A^{-2}}{1+\sigma_A^{-2}} \theta_t + \frac{1}{1+\sigma_A^{-2}} r_{j_0,t-1} + noise$$

where the noise term has variance of order $1/n$ and depends on signal noise, variation in $r_{j,t}$, and variation in weights. These bounds on the noise term follow from the assumption

[48]We use the notation $\Omega(n)$ to mean greater than Cn for some constant $C > 0$ when n is large.

that the covariance matrix of the $r_{i,t}$ is in a neighborhood $\mathcal{W}_{\psi,\zeta}$ and our analysis of variation in weights.Similarly

$$\frac{1}{|N_{i,B}|}\sum_{j\in N_{i,B}} a_{j,t-1} = \frac{\sigma_B^{-2}}{1+\sigma_B^{-2}}\theta_t + \frac{1}{1+\sigma_B^{-2}} r_{j_0,t-1} + noise$$

where the noise term has the same properties. Because $\sigma_A^2 \neq \sigma_B^2$, we can write θ_t as a linear combination of these two averages with coefficients independent of n up to a noise term of order $1/n$. We can choose β large enough such that this noise term has variance most β/n for all n sufficiently large. This completes the Proof of Lemma 1.

C.3.4. *Step 4:* We now give the two-step version of Lemma 3.

Lemma 5. *Suppose that in period t the matrix $A = A_t$ of covariances of social signals satisfies $A \in \mathcal{W}_{\psi,\zeta}$ and all agents are optimizing in periods $t+1$ and $t+2$. Then there is a γ so that for all n sufficiently large,*

$$\frac{W_{ij,t+2}}{W_{i'j',t+2}} \in \left[1-\frac{\gamma}{n}, 1+\frac{\gamma}{n}\right].$$

whenever i and i' have the same network and signal types and j and j' have the same network and signal types.

Given what we established about covariances in Step 3, the lemma follows by the same argument as the proof of Lemma 3.

Step 5: Now that Lemma 5 is proved, we can apply it to show that

$$\widetilde{\Phi}^2(\mathcal{W}_{\psi,\zeta}) \subset \mathcal{W}_{\psi,\zeta}.$$

We will do this by first writing the time-$(t+2)$ behavior in terms of agents' time-t observations (Step 5(a)), which comes from applying $\widetilde{\Phi}$ twice. This gives a formula that can be used for bounding the covariances[49] of time-$(t+2)$ actions in terms of covariances of time-t actions. Step 5(b) then applies this formula to show we can take ζ_{ij} and ζ_i to be sufficiently small. (Recall the notation introduced in Section C.1 above.) We split our expression for $r_{i,t+2}$ into several groups of terms and show that the contribution of each group of terms depends only on agents' types up to a small noise term. Step 5(c) notes that we can also take $\psi_{kk'}$ and ψ_k to be sufficiently small.

[49]We take this term to refer to variances, as well.

Step 5(a): We calculate:

$$\begin{aligned} r_{i,t+2} &= \sum_j \frac{W_{ij,t+2}}{1-w^s_{i,t+2}}\rho a_{j,t+1} \\ &= \rho(\sum_j \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} w^s_{j,t+1}s_{j,t+1} + \sum_{j,j'} \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} W_{jj',t+1}\rho a_{j',t}) \\ &= \rho(\sum_j \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} w^s_{j,t+1}s_{j,t+1} + \rho(\sum_{j,j'} \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} W_{jj',t+1}w^s_{j',t}s_{j',t} \\ &\quad + \sum_{j,j'} \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} W_{jj',t+1}(1-w^s_{j',t})r_{j',t})). \end{aligned}$$

Let $c_{ij',t}$ be the coefficient on $r_{j',t}$ in this expansion of $r_{i,t+2}$. Explicitly,

$$c_{ij',t} = \sum_j \frac{W_{ij,t+2}}{1-w^s_{i,t+2}} W_{jj',t+1}(1-w^s_{j',t}). \tag{C.3}$$

The coefficient $c_{ij',t}$ adds up the influence of $r_{j',t}$ on $r_{i,t+2}$ over all paths of length two.

First, we establish a lemma about how much these weights vary.

Lemma 6. *There exists γ such that for n sufficiently large, when i and i' have the same network types and j' and j'' have the same network and signal types, the ratio $c_{ij',t}/c_{i'j'',t}$ is in $[1-\gamma/n, 1+\gamma/n]$.*

Proof. Fix i and j'. For each network type k'' and signal type s, consider the number of agents j of network type k'' and signal type s who are observed by i and who observe j'. This number varies by at most a factor ζ^2 as we change i and j', preserving signal and network types. For each such j, the contribution of that agent's action to $c_{ij',t}$ is (recalling (C.3))

$$\frac{W_{ij,t+2}}{1-w^s_{i,t+2}} W_{jj',t+1}(1-w^s_{j',t}).$$

By applying Lemma 3 repeatedly, we can choose γ_1 such that each of these contributions varies by at most a factor of γ_1/n as we change i in G_k and j' in $G_{k'}$. Thus, $c_{ij',t}$ is a sum of terms which vary by at most a multiplicative factor of γ_1/n as we change i and j' preserving signal and network types. If we can show that the sum of the absolute values of these terms is bounded, then it will follow that $c_{ij',t}$ varies by at most a multiplicative factor of γ/n for some n. This bound on the sum of absolute values follows from the calculation of weights in the proof of Lemma 3. □

Step 5(b): We first show that fixing the values of $\psi_{kk'}$ and ψ_k in period t, the variation in the covariances $\mathrm{Cov}(r_{i,t+2}-\theta_{t+1,}r_{i',t+2}-\theta_{t+1})$ of these terms as we vary i and i' over network types is not larger than ζ. From the formula above, we observe that we can decompose $r_{i,t+2} - \theta_{t+1}$ as a linear combination of three mutually independent groups of terms:

(i) signal error terms $\eta_{j,t+1}$ and $\eta_{j',t}$;

(ii) the errors $r_{j',t} - \theta_t$ in the social signals from period t; and

(iii) changes in state $\nu_t$and ν_{t+1} between periods t and $t+2$.

Note that the terms $r_{j',t} - \theta_t$ are linear combinations of older signal errors and changes in the state. We bound each of the three groups in turn:

(i) Signal Errors: We first consider the contribution of signal errors. When i and i' are distinct, the number of such terms is close to its expected value because we are conditioning on the events E_2 and E_4 defined in Section C.1. Moreover the weights are close to their expected values by Step 2, so the variation is bounded suitably. When i and i' are equal, we use the facts that the weights are close to their expected values and the variance of an average of $\Omega(n)$ signals is small.

(ii) Social Signals: We now consider terms $r_{j',t} - \theta_t$, which correspond to the third summand in our expression for $r_{i,t+2}$. Since we will analyze the weight on ν_t below, it is sufficient to study the terms $r_{j',t} - \theta_{t-1}$.

By Lemma 6, the coefficients placed on $r_{j',t}$ by i and on $r_{j'',t}$ by i' vary by a factor of at most $2\gamma/n$. Moreover, the absolute value of each of these covariances is bounded above by ψ and the variation in these terms is bounded above by ζ. We conclude that the variation from these terms has order $1/n^2$.

(iii) Innovations: Finally, we consider the contribution of the innovations ν_t and ν_{t+1}. We treat ν_{t+1} first. We must show that any two agents of the same types place the same weight on the innovation ν_{t+1} (up to an error of order $\frac{1}{n^2}$). This will imply that the contributions of timing to the covariances $\mathrm{Cov}(r_{i,t+2} - \theta_{t+1}, r_{i',t+2} - \theta_{t+1})$ can be expressed as a term that can be included in the relevant $\psi_{kk'}$ and a lower-order term which can be included in $\zeta_{ii'}$.

The weight an agent places on ν_{t+1} is equal to the weight she places on signals from period $t+1$. So this is equivalent to showing that the total weight

$$\rho \sum_j \frac{W_{ij,t+2}}{1 - w^s_{i,t+2}} w^s_{j,t+1}$$

agent i places on period $t+1$ depends only on the network type k of agent i and $O(1/n^2)$ terms. We will first show the average weight placed on time-$(t+1)$ signals by agents of each signal type depends only on k. We will then show that the total weights on agents of each signal type do not depend on n.

Suppose for simplicity here that there are two signal types A and B; the general case is the same. We can split the sum from the previous paragraph into the subgroups of agents with signal types A and B:

$$\rho \sum_{j:\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{1-w_{i,t+2}^s} w_{j,t+1}^s + \rho \sum_{j:\sigma_j^2=\sigma_B^2} \frac{W_{ij,t+2}}{1-w_{i,t+2}^s} w_{j,t+1}^s.$$

Letting $W_i^A = \sum_{\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{1-w_{i,t+2}^s}$ be the total weight placed on agents with signal type A and similarly for signal type B, we can rewrite this as:

$$W_i^A \rho \sum_{j:\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)} w_{j,t+1}^s + W_i^B \rho \sum_{j:\sigma_j^2=\sigma_B^2} \frac{W_{ij,t+2}}{W_i^B(1-w_{i,t+2}^s)} w_{j,t+1}^s.$$

The coefficients $\frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)}$ in the first sum now sum to one, and similarly for the second. We want to check that the first sum $\sum_{j:\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)} w_{j,t+1}^s$ does not depend on k, and the second sum is similar.

For each j in group A,

$$w_{j,t+1}^s = \frac{\sigma_A^{-2}}{\sigma_A^{-2} + (\rho^2 \kappa_{j,t+1} + 1)^{-1}},$$

where we define $\kappa_{j,t+1}^2 = \mathrm{Var}(r_{j,t+1} - \theta_t)$ to be the error variance of the social signal. Because $\kappa_{j,t+1}$ is close to zero, we can approximate $w_{j,t+1}^s$ locally as a linear function $\mu_1 \kappa_{j,t+1} + \mu_2$ where $\mu_1 < 1$ (up to order $\frac{1}{n^2}$ terms).

So we can write the sum of interest as

$$\sum_{j:\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)} (\mu_1 \sum_{j',j''} W_{jj',t+1} W_{jj'',t+1} (\rho^2 \boldsymbol{V}_{j'j'',t} + 1) + \mu_2).$$

By Lemma 3, the weights vary by at most a multiplicative factor contained in $[1-\gamma/n, 1+\gamma/n]$. The number of paths from i to j' passing through agents of any network type k'' and any signal type is close to its expected value (which depends only on i's network type), and the weight on each path depends only on the types involved up to a factor in $[1-\gamma/n, 1+\gamma/n]$. The variation in $\boldsymbol{V}_{j'j'',t}$ consists of terms of the form $\psi_{k'k''}$, $\psi_{k'}$, and $\zeta_{j'j''}$, all of which are $O(1/n)$, and terms from signal errors $\eta_{j',t}$. The signal errors only contribute when $j = j'$, and so only contribute to a fraction of the summands of order $1/n$. So we can conclude the total variation in this sum as we change i within the network type k has order $1/n^2$.

Now that we know each the average weight on private signals of the observed agents of each signal type depends only on k, it remains to check that W_i^A and W_i^B only depend on

k. The coefficients W_i^A and W_i^B are the optimal weights on the group averages

$$\sum_{j:\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)}\rho a_{j,t+1} \text{ and } \sum_{j:\sigma_j^2=\sigma_B^2} \frac{W_{ij,t+2}}{W_i^B(1-w_{i,t+2}^s)}\rho a_{j,t+1},$$

so we need to show that the variances and covariance of these two terms depend only on k. We check the variance of the first sum: we can expand

$$\sum_{\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)}\rho a_{j,t+1} = \sum_{\sigma_j^2=\sigma_A^2} \frac{W_{ij,t+2}}{W_i^A(1-w_{i,t+2}^s)}\rho(w_{j,t+1}^s s_{j,t+1} + (1-w_{j,t+1}^s)r_{j,t+1}).$$

We can again bound the signal errors and social signals as in the previous parts of this proof, and show that the variance of this term depends only on k and $O(1/n^2)$ terms. The second variance and covariance are similar, so W_i^A and W_i^B depend only on k and $O(1/n^2)$ terms.

This takes care of the innovation ν_{t+1}. Because we have included any innovations prior to ν_t in the social signals $r_{j',t}$, to complete Step 5(b) we need only show the weight on ν_t depends only on the network type k of an agent.

The analysis is a simpler version of the analysis of the weight on ν_{t+1}. It is sufficient to show the total weight placed on period t social signals depends only on the network type of k of an agent i. This weight is equal to

$$\rho^2 \sum_{j,j'} \frac{W_{ij,t+2}}{1-w_{i,t+2}^s} \cdot W_{jj',t+1} \cdot (1-w_{j',t}^s).$$

As in the ν_{t+1} case, we can approximate $(1-w_{j',t}^s)$ as a linear function of $\kappa_{j',t}$ up to $O(1/n^2)$ terms. Because the number of paths to each agent j' though a given type and the weights on each such path cannot vary too much within types, the same argument shows that this sum depends only on k and $O(1/n^2)$ terms.

Step 5(b) is complete.

Step 5(c): The final step is to verify that we can take $\psi_{kk'}$ and ψ_k to be smaller than ψ. It is sufficient to show that the variance $\mathrm{Var}(r_{i,t+2} - \theta_{t+1})$ of each social signal about θ_{t+1} is at most ψ. The proof is the same as in Step 2(b).

Appendix D. Model with a starting time

In introducing the model (Section 2), we made the set of time indices $\mathcal{T}$ equal to $\mathbb{Z}$, the set of all integers. Here we study the variant with an initial time period, $t = 0$: thus, we take $\mathcal{T}$ to be $\mathbb{Z}_{\geq 0}$, the nonnegative integers.

We let θ_0 be drawn according to the stationary distribution of the state process: $\theta_0 \sim \mathcal{N}\left(0, \frac{1}{1-\rho}\right)$. After this, the state random variables θ_t satisfy the AR(1) evolution

$$\theta_{t+1} = \rho\theta_t + \nu_{t+1},$$

where ρ is a constant with $0 < |\rho| < 1$ and $\nu_{t+1} \sim \mathcal{N}(0, \sigma_\nu^2)$ are independent innovations. Actions, payoffs, signals, and observations are the same as in the main model, with the obvious modification that in the initial periods, $t < m$, information sets are smaller as there are not yet prior actions to observe.[50] To save on notation, we write actions as if agents had an improper prior, understanding that the adjustment for actions taken under the natural prior $\theta_t \sim \mathcal{N}\left(0, \frac{1}{1-\rho}\right)$ is immediate.

In this model, there is a straightforward prediction of behavior. A Nash equilibrium here refers to an equilibrium of the game involving all agents (i, t) for all time indices in $\mathcal{T}$.

Fact 2. *In the model with $\mathcal{T} = \mathbb{Z}_{\geq 0}$, there is a unique Nash equilibrium, and it is in linear strategies. The initial generation ($t = 0$) plays a linear strategy based on private signals only. In any period $t > 0$, given linear strategies from prior periods, players' best responses are linear. For time periods $t > m$, we have*

$$\boldsymbol{V}_t = \Phi(\boldsymbol{V}_{t-1}).$$

This fact follows from the observation that the initial ($t = 0$) generation faces a problem of forming a conditional expectation of a Gaussian state based on Gaussian signals, so their optimal strategies are linear. From then on, the analysis of Section 3.1 characterizes best-response behavior inductively. Note that for arbitrary environments, the fact does not imply that $\boldsymbol{V}_t$ must converge.

Our main purpose in this section is to give analogues of the main results on learning in large networks. We use the same definition of an environment—in terms of the distribution of networks and signals—as in Section 4.1. For simplicity, we work with $m = 1$, though the arguments for our positive result extend straightforwardly.

The analogue of Theorem 1 is:

Theorem 3. *Consider the $\mathcal{T} = \mathbb{Z}_{\geq 0}$ model. If an environment satisfies signal diversity, there is $C > 0$ such that asymptotically almost surely $\hat{\kappa}_{i,t}^2 < C/n$ for all i at all times $t \geq 1$ in the unique Nash equilibrium.*

In particular, this implies that the covariance matrix in each period $t \geq 1$ is very close (in the Euclidean norm) to the good-learning equilibrium from Theorem 1. We sketch the

[50]The actions for $t < 0$ can be set to arbitrary (commonly known) constants.

proof, which uses the material we developed in Appendix C. We define A_t as in that proof (Section C.1). Take a $\beta > 0$, to be specified later, and consider

$$\overline{\mathcal{W}} = \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}} \cup \widetilde{\Phi}(\mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}).$$

First, for large enough β, we have that $A_1 \in \overline{\mathcal{W}}$: In the unique Nash equilibrium, at $t = 1$, agents simply take weighted averages of their neighbors' signals, weighted by their precisions. So $A_1 \in \overline{\mathcal{W}}$ by the central limit theorem for β sufficiently large. Second, we use the previously established fact (recall Section C.2.2) that $\widetilde{\Phi}(\overline{\mathcal{W}}) \subset \overline{\mathcal{W}}$ to deduce that $A_t \in \overline{\mathcal{W}}$ at all future times. Finally, we observe that $\overline{\mathcal{W}} \subseteq \mathcal{W}_{\frac{\beta}{n},\frac{1}{n}}$ by construction.

Without signal diversity, bad learning can occur forever, in the unique equilibrium. The analogue of Proposition 2 is immediate. In graphs with symmetric neighbors, Φ is a contraction when $m = 1$. So iteration of it arrives at the unique fixed point, and thus a learning outcome far from the benchmark.

Appendix E. Simulations on Random Networks

In Section 7.1, we found that private signal diversity substantially improved learning outcomes. To give some intuition for the speed of convergence for our main results, we now provide an illustration via simulations on Erdos-Renyi random networks. We find that signal diversity cuts average social signal variances by factors of two or more in random networks with expected degrees of less than or equal to 100.

We consider directed random networks with n agents. Each edge is drawn independently with probability $p = .25$. For each network size and each signal variances, we simulate 100 random networks with $\rho = 0.9$ and compute the average equilibrium social signal variance across all agents and all networks. We consider a homogeneous treatment, in which all private signal variances are 2, and a heterogeneous treatment, in which half of private signal variances are $\frac{3}{2}$ and half of private signal variances are 3. Private signal assignments are independent of the network. The two treatments have the same average private signal precision, as in Section 7.1.

Figure E.1. Social Signal Variance In Indian Villages

Network Size (n)	Homogeneous	Heterogeneous
100	0.496	0.309
200	0.490	0.221
400	0.487	0.140

Average social signal variances across all villages with homogeneous and heterogeneous private signals for Erdos-Renyi networks of size $n = 100, 200$, and 400.

Results are displayed in Figure E.1. When the expected degree is 50, the average social signal error in the homogeneous case is more than twice as large as in the heterogeneous case. When the expected degree is 100, the average social signal error in the homogeneous case is more than three times as large as in the heterogeneous case.

Appendix F. Remaining proofs (online appendix)

F.1. **Proof of Corollary 1.** Suppose the social influence

$$SI(i) = \sum_{j \in N} \sum_{k=0}^{\infty} (\rho \hat{W})^k_{ji} \hat{w}^s_i = [\mathbf{1}'(I - \rho \hat{W})^{-1}]_i \hat{w}^s_i$$

does not converge for some i. Then in particular, there exists j such that $\sum_{k=0}^{\infty} (\rho \hat{W})^k_{ji} \hat{w}^s_i$ does not converge. We can write

$$a_{j,t} = \sum_{\tau=0}^{\infty} \sum_{j' \in N} (\rho \hat{W})^{\tau}_{jj'} \hat{w}^s_{j'} s_{j',t-\tau}.$$

This expression is the sum of

$$\sum_{\tau=0}^{\infty} (\rho \hat{W})^{\tau}_{ji} \hat{w}^s_i \eta_{i,t-\tau}$$

and independent terms corresponding to signal errors of agents other than i and changes in the state. Because $\sum_{\tau=0}^{\infty} (\rho \hat{W})^{\tau}_{ji} \hat{w}^s_i$ does not converge, the payoff to action $a_{j,t}$ must therefore be $-\infty$. But we showed in the proof of Proposition 1 that agent j's equilibrium payoff is at least $-\sigma_j^2$, which gives a contradiction.

Given convergence, the expression for $SI(i)$ follows from the identity $(I - M)^{-1} = \sum_{k=0}^{\infty} M^k$.

F.2. **Proof of Proposition 2.** We first check there is a unique equilibrium and then prove the remainder of Proposition 2.

Lemma 7. *Suppose G has symmetric neighbors. Then there is a unique equilibrium.*

Proof of Lemma 7. We will show that when the network satisfies the condition in the proposition statement, Φ induces a contraction on a suitable space. For each agent, we can consider the variance of the best estimator for yesterday's state based on observed actions. These variances are tractable because they satisfy the envelope theorem. Moreover, the space of these variances is a sufficient statistic for determining all agent strategies and action variances.

Let $r_{i,t}$ be i's *social signal*—the best estimator of θ_{t-1} based on the period $t-1$ actions of agents in N_i—and let $\kappa_{i,t}^2$ be the variance of $r_{i,t} - \theta_{t-1}$.

We claim that Φ induces a map $\widetilde{\Phi}$ on the space of variances $\kappa_{i,t}^2$, which we denote $\widetilde{\mathcal{V}}$. We must check the period t variances $(\kappa_{i,t}^2)_i$ uniquely determine all period $t+1$ variances $(\kappa_{i,t+1}^2)_i$: The variance $\boldsymbol{V}_{ii,t}$ of agent i's action, as well as the covariances $\boldsymbol{V}_{ii',t}$ of all pairs of agents i, i' with $N_i = N_{i'}$, are determined by $\kappa_{i,t}^2$. Moreover, by the condition on our network, these variances and covariances determine all agents' strategies in period $t+1$, and this is enough to pin down all period $t+1$ variances $\kappa_{i,t+1}^2$.

The proof proceeds by showing $\widetilde{\Phi}$ is a contraction on $\widetilde{\mathcal{V}}$ in the sup norm.

For each agent j, we have $N_i = N_{i'}$ for all $i, i' \in N_j$. So the period t actions of an agent i' in N_j are

$$a_{i',t} = \frac{(\rho^2\kappa_{i,t}^2+1)^{-1}}{\sigma_{i'}^{-2}+(\rho^2\kappa_{i,t}^2+1)^{-1}} \cdot r_{i,t} + \frac{\sigma_{i'}^{-2}}{\sigma_{i'}^{-2}+(\rho^2\kappa_{i,t}^2+1)^{-1}} \cdot s_{i',t} \tag{F.1}$$

where $s_{i',t}$ is agent (i')'s signal in period t and $r_{i,t}$ the social signal of i (the same one that i' has). It follows from this formula that each action observed by j is a linear combination of a private signal and a *common* estimator $r_{i,t}$, with positive coefficients which sum to one. For simplicity we write

$$a_{i',t} = b_0 \cdot r_{i,t} + b_{i'} \cdot s_{i',t} \tag{F.2}$$

(where b_0 and $b_{i'}$ depend on i' and t, but we omit these subscripts). We will use the facts $0 < b_0 < 1$ and $0 < b_{i'} < 1$.

We are interested in how $\kappa_{j,t+1}^2 = \mathrm{Var}(r_{j,t+1} - \theta_t)$ depends on $\kappa_{i,t}^2 = \mathrm{Var}(r_{i,t} - \theta_{t-1})$. The estimator $r_{j,t+1}$ is a linear combination of observed actions $a_{i',t}$, and therefore can be expanded as a linear combination of signals $s_{i',t}$ and the estimator $r_{i,t}$. We can write

$$r_{j,t+1} = c_0 \cdot (\rho r_{i,t}) + \sum_{i'} c_{i'} s_{i',t} \tag{F.3}$$

and therefore (taking variances of both sides)

$$\begin{aligned} \kappa_{j,t+1}^2 = \mathrm{Var}(r_{j,t+1} - \theta_t) &= c_0^2 \mathrm{Var}(\rho r_{i,t} - \theta_t) + \sum_{i'} c_{i'}^2 \sigma_{i'}^2 \\ &= c_0^2(\rho^2\kappa_{i,t}^2 + 1) + \sum_{i'} c_{i'}^2 \sigma_{i'}^2 \end{aligned}$$

The desired result, that $\widetilde{\Phi}$ is a contraction, will follow if we can show that the derivative $\frac{d\kappa_{j,t+1}^2}{d\kappa_{i,t}^2} = c_0^2\rho^2 \in [0, \delta]$ for some $\delta < 1$. By the envelope theorem, when calculating this derivative, we can assume that the weights placed on actions $a_{i',t}$ by the estimator $r_{j,t}$ do not change as we vary $\kappa_{i,t}^2$, and therefore c_0 and the $c_{i'}$ above do not change. So it is enough to show the coefficient c_0 is in $[0, 1]$. $\square$

The intuition for the lower bound is that *anti-imitation* (agents placing negative weights on observed actions) only occurs if observed actions put too much weight on public information. But if $c_0 < 0$, then the weight on public information is actually negative so there is no reason to anti-imitate. This is formalized in the following lemma.

Lemma 8. *Suppose j has symmetric neighbors. Then agent j's social signal places non-negative weight on a neighbor$i$'s social signal from the previous period, i.e., $c_0 \geq 0$.*

Proof. To check this formally, suppose that c_0 is negative. Then the social signal $r_{j,t+1}$ puts negative weight on some observed action—say the action $a_{k,t}$ of agent k. We want to check that the covariance of $r_{j,t+1} - \theta_t$ and $a_{k,t} - \theta_t$ is negative. Using (F.2) and (F.3), we compute that

$$\begin{aligned}\mathrm{Cov}(r_{j,t+1} - \theta_t, a_{k,t} - \theta_t) &= \mathrm{Cov}\left(c_0(\rho r_{i,t} - \theta_t) + \sum_{i' \in N_j} c_{i'}(s_{i',t} - \theta_t)), b_0(\rho r_{i,t} - \theta_t) + b_k(s_{k,t} - \theta_t)\right) \\ &= c_0 b_0 \mathrm{Var}(\rho r_{i,t} - \theta_t) + c_k b_k \mathrm{Var}(s_{k,t} - \theta_t)\end{aligned}$$

because all distinct summands above are mutually independent. We have $b_0, b_k > 0$, while $c_0 < 0$ by assumption and $c_k < 0$ because the estimator $r_{j,t+1}$ puts negative weight on $a_{k,t}$. So the expression above is negative. Therefore, it follows from the usual Gaussian Bayesian updating formula that the best estimator of θ_t given $r_{j,t+1}$ and $a_{k,t}$ puts positive weight on $a_{k,t}$. However, this is a contradiction: the best estimator of θ_t given $r_{j,t+1}$ and $a_{k,t}$ is simply $r_{j,t+1}$, because $r_{j,t+1}$ was defined as the best estimator of θ_t given observations that included $a_{k,t}$. □

We now complete the proof of Lemma 7.

Proof. Now, for the upper bound $c_0 \leq 1$, the idea is that $r_{j,t+1}$ puts more weight on agents with better signals while these agents put little weight on public information, which keeps the overall weight on public information from growing too large.

Note that $r_{j,t+1}$ is a linear combination of actions $\rho a_{i',t}$ for $i' \in N_j$, with coefficients summing to 1. The only way the coefficient on $\rho r_{i,t}$ in $r_{j,t+1}$ could be at least 1 would be if some of these coefficients on $\rho a_{i',t}$ were negative and the estimator $r_{j,t+1}$ placed greater weight on actions $a_{i',t}$ which placed more weight on $r_{i,t}$.

Applying the formula (F.1) for $a_{i',t}$, we see that the coefficient b_0 on $\rho r_{i,t}$ is less than 1 and increasing in $\sigma_{i'}$. On the other hand, it is clear that the weight on $a_{i',t}$ in the social signal $r_{j,t+1}$ is decreasing in $\sigma_{i'}$: more weight should be put on more precise individuals. So in fact the estimator $r_{j,t+1}$ places less weight on actions $a_{i',t}$ which placed more weight on $r_{i,t}$.

Moreover, the coefficients placed on private signals are bounded below by a positive constant when we restrict to covariances in the image of $\widetilde{\Phi}$ (because all covariances are bounded as in the proof of Proposition 1). Therefore, each agent $i' \in N_j$ places weight at most one on the estimator $\rho r_{i,t-1}$. Agent j's social signal $r_{j,t+1}$ is a sum of these agents' actions with coefficients summing to 1 and satisfying the monotonicity property above. We conclude that the coefficient on $\rho r_{i,t}$ in the expression for $r_{j,t+1}$ is at most one. □

This completes the proof of Lemma 7. We now prove Proposition 2.

Proof of Proposition 2. By Lemma 7 there is a unique equilibrium on any network G with symmetric neighbors. Let $\varepsilon > 0$.

Consider any agent i. Her neighbors have the same private signal qualities and the same neighborhoods (by the symmetric neighbors assumption). So there exists an equilibrium where for all i, the actions of agent i's neighbors are exchangeable. By uniqueness, this in fact holds at the sole equilibrium.

So agent i's social signal is an average of her neighbors' actions:

$$r_{i,t} = \frac{1}{|N_i|} \sum_{j \in N_i} a_{j,t-1}.$$

Suppose the ε-aggregation benchmark is achieved. Then all agents must place weight at least $\frac{(1+\varepsilon)^{-1}}{(1+\varepsilon)^{-1}+\sigma^{-2}}$ on their social signals. So at time t, the social signal $r_{i,t}$ places weight at least $\frac{(1+\varepsilon)^{-1}}{(1+\varepsilon)^{-1}+\sigma^{-2}}$ on signals from at least two periods ago. Since the variance of any linear combination of such signals is at least $1+\rho$, for ε sufficiently small the social signal $r_{i,t}$ is bounded away from a perfect estimate of θ_{t-1}. This gives a contradiction. □

F.3. **Proof of Corollary 2.** Consider a complete graph in which all agents have signal variance σ^2 and memory $m = 1$. By Proposition 2, as n grows large the variances of all agents converge to $A > (1+\sigma^{-2})^{-1}$.

Choose σ^2 large enough such that $A > 1$. To see that we can do this, note that as σ^2 grows large, the weight each agent places on their private signal vanishes. So the weight on signals from at least k periods ago approaches one for any k. Taking σ^2 such that this holds for k sufficiently large, we have$A > 1$.

Now suppose that we increase σ_1^2 to ∞. Then $a_{1,t} = r_{1,t}$ in each period, so all agents can infer all private signals from the previous period. As n grows large, the variance of agent 1 converges to 1 and the variances of all other agents converge to $(1+\sigma^{-2})^{-1}$. By our choice of σ^2, this gives a Pareto improvement. We can see by continuity that the same argument holds for σ_1^2 finite but sufficiently large.

F.4. **Proof of Theorem 2.** Suppose that all private signals have variance $\sigma^2 > 0$. Fix a sequence of networks G_n and an equilibrium on each G_n. We will show that given any constant $C > 0$ and any sequence of equilibria, the fraction of agents i such that

$$\hat{\kappa}_i^2 \leq \frac{C}{n}$$

is bounded away from one.

We first prove the result in the case $m = 1$. For each n, let $\mathscr{G}_n$ be the set of agents i satisfying

$$\hat{\kappa}_i^2 \leq \frac{C}{n},$$

i.e., the set of agents who do learn well. Assume for the sake of contradiction that $\frac{|\mathscr{G}_n|}{n} \to 1$ as $n \to \infty$ along some subsequence and pass to that subsequence.

For each j, we can express the action $a_{j,t}$ as a weighted sum of innovations and signal errors,[51] with all terms on the right-hand side conditionally independent:

$$a_{j,t} = \theta_t - \sum_{l=0}^{\infty} w_{j,t}(\nu_{t-l})(\rho^l \nu_{t-l}) + \sum_{l,j'} w_{j,t}(\eta_{j',t-l})(\rho^l \eta_{j',t-l}).$$

This expression is unique.

Lemma 9. *For all $j \in \mathscr{G}_n$ we must have*

$$w_{j,t}(\nu_t) \in (\frac{1}{\sigma^{-2}+1} - \frac{C'}{n}, \frac{1}{\sigma^{-2}+1})$$

for some $C' > 0$ (independent of j and n).

Proof. By the standard updating formula, the optimal weight $w_{j,t}(\nu_t)$ is $\frac{(\rho^2\kappa_{j,t}^2+1)^{-1}}{(\rho^2\kappa_{j,t}^2+1)^{-1}+\sigma^{-2}}$, where $\kappa_{j,t}^2$ is the variance of the best estimator of θ_{t-1} based on (j,t)'s social observations. The upper bound follows because this is minimized when $\kappa_{j,t}^2 = 0$. For the lower bound,

$$\begin{aligned}
w_{j,t}(\nu_t) &= \frac{(\rho^2\kappa_{j,t}^2+1)^{-1}}{(\rho^2\kappa_{j,t}^2+1)^{-1}+\sigma^{-2}} \\
&= \frac{1}{(1+\sigma^{-2}) + \sigma^{-2}\rho^2\kappa_{j,t}^2} \\
&= \frac{1}{1+\sigma^{-2}} - \frac{\sigma^{-2}\rho^2}{(1+\sigma^{-2})^2}\kappa_{j,t}^2 + O(\kappa_{j,t}^4).
\end{aligned}$$

[51]To simplify calculations, we write this expression with a negative coefficient on the first sum so that the terms $w_{j,t}(\nu_{t-l})$ are positive. The weight that j places on ν_{t-l} is in fact $-w_{j,t}(\nu_{t-l})$.

For $\kappa_{j,t}^2$ in any neighborhood of zero, we can choose C' such that the non-constant terms in the final expression are bounded below by $-C'\kappa_{j,t}^2$. Since by assumption we have $\kappa_{j,t}^2 \leq \frac{C}{n}$, the lemma follows. $\square$

Now consider the action of agent $i \in \mathscr{G}_n$ in period $t+1$ observing N_i. Since

$$\kappa_{i,t+1}^2 \leq \frac{C}{n},$$

the weight on the innovation from the *previous* period satisfies

$$(w_{i,t+1}(\nu_t)\rho)^2 \leq \frac{C}{n}. \tag{F.4}$$

On the other hand, we can express this weight in terms of neighbors' weights as

$$w_{i,t+1}(\nu_t) = \sum_j \rho w_{ij,t+1} w_{j,t}(\nu_t).$$

Here $\sum_j w_{ij,t+1} = w_{i,t}(\nu_{t-1})$ converges to $\frac{1}{1+\sigma^{-2}}$ at rate $O(\frac{1}{n})$ as $n \to \infty$ by Lemma 9. We will show that if this weight $w_{i,t+1}(\nu_t)$ vanishes, then the contribution of private signal errors to $\kappa_{i,t+1}$ must be larger than $O(\frac{1}{n})$.

We can split this summation as

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + \rho \sum_{j \notin \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t).$$

We now consider two cases, depending on whether $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$, i.e., whether the sum of the absoulte values of the weights on agents outside $\mathscr{G}_n$ is vanishing.

Case 1: $\liminf_n \sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| = 0$. We can pass to a subsequence along which $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$.

We claim that it follows from the bounds on $w_{j,t}(\nu_t)$ in Lemma 9 that this can only occur if $\sum_j |w_{ij,t+1}| \to \infty$. If $\sum_j |w_{ij,t+1}|$ is bounded,

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + \rho \sum_{j \notin \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + o(1).$$

The second equality holds because $\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| \to 0$ and $w_{j,t}(\nu_t) \in [0,1]$ for all j. Therefore,

$$w_{i,t+1}(\nu_t) = \rho \sum_{j \in \mathscr{G}_n} w_{ij,t+1} w_{j,t}(\nu_t) + o(1) \geq \left(\rho \frac{1}{1+\sigma^{-2}} \frac{\sigma^{-2}}{\sigma^{-2}+1} - o\left(1\right)\right) + o(1),$$

and the right-hand side is non-vanishing. Here the first term on the right-hand side is the limit of the sum if all of the terms $w_{j,t}(\nu_t)$ were equal to the upper bound $\frac{\sigma^{-2}}{\sigma^{-2}+1}$. The first $o(1)$ error term corresponds to the variation in $w_{j,t}(\nu_t)$ across j, which is $O(\frac{1}{n})$ by Lemma 9 and has bounded coefficients. Thus $w_{i,t+1}(\nu_t)$ is non-vanishing, but this contradicts the inequality (F.4). We have proven the claim.

The contribution to $\kappa_{i,t+1}^2$ from signal errors $\eta_{j,t}$ is $\sum_j |w_{ij,t+1}|^2 (w_{j,t}^s)^2 \sigma^2$. Since $w_{j,t}^s = 1 - w_{j,t}(\nu_t)$ converge uniformly to a constant $\frac{\sigma^{-2}}{\sigma^{-2}+1}$, we can bound this contribution below by an expression that is proportional to

$$\sum_j |w_{ij,t+1}|^2.$$

Applying the standard bound $\|v\|_1 \leq \sqrt{n}\|v\|_2$ on L^p norms,

$$\sum_j |w_{ij,t+1}|^2 \geq \frac{1}{n}(\sum_j |w_{ij,t+1}|)^2.$$

The right-hand side of this inequality grows at a rate faster than $\frac{1}{n}$ by the claim $\sum_j |w_{ij,t+1}| \to \infty$, and so the social signal error grows at a rate faster than $\frac{1}{n}$. This gives a contradiction.

Case 2: $\liminf_n \sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}| > 0$.

As in Case 1, the contribution to signal errors from neighbors $j \notin \mathscr{G}_n$ is proportional to

$$\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}|^2.$$

By the same standard bound on L^p norms,

$$\sum_{j \notin \mathscr{G}_n} |w_{ij,t+1}|^2 \geq \frac{1}{n - |\mathscr{G}_n|}(\sum_{j \in \mathscr{G}_n} |w_{ij,t+1}|)^2.$$

By assumption, the cardinality $n - |\mathscr{G}_n|$ of the complement of $\mathscr{G}_n$ is $o(n)$ and $(\sum_{j \in \mathscr{G}_n} |w_{ij,t+1}|)^2$ is non-vanishing. So the right-hand side grows at a rate faster than $\frac{1}{n}$. Thus the social signal error grows at a rate faster than $\frac{1}{n}$, which again gives a contradiction.

Now, suppose $m \geq 1$ is arbitrary. As before, for each agent (j,t),we can write:

$$a_{j,t} = \theta_t - \sum_{l=0}^{\infty} w_{j,t}(\nu_{t-l})(\rho^l \nu_{t-l}) + \sum_{l,j'} w_{j,t}(\eta_{j',t-l})(\rho^l \eta_{j',t-l}).$$

For each n, let $\mathscr{G}_n$ be the set of agents i satisfying

$$\hat{\kappa}_i^2 \leq \frac{C}{n}.$$

Suppose $\limsup_n |\mathscr{G}_n|/n = 1$. Passing to a subsequence, we can assume that $\lim_n |\mathscr{G}_n|/n = 1$, i.e., the fraction of agents in $\mathscr{G}_n$ converges to one.

Consider an agent (i,t) with $i \in \mathscr{G}_n$, who observes neighbors' actions in periods $t-1,\ldots,t-m$. For each $1 \le l \le m$, we will write $w_{(i,t),(j,t-l)}$ for the weight that agent (i,t) places on the action of agent $(j,t-l)$. By the same argument as in Case 2 of the $m=1$ proof above, $\liminf_n \sum_{j\notin\mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$ for each l (since the fraction of agents outside $\mathscr{G}_n$ is vanishing). Passing to a subsequence, we can assume that $\lim_n \sum_{j\notin\mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$.

We can express agent (i,t)'s action:

$$a_{i,t} = \sum_{1\le l\le m}\left(\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l a_{j,t-l} + \sum_{j\notin\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l a_{j,t-l}\right).$$

We will show that this expression places non-vanishing weight on the innovation ν_{t-l} for some $l \ge 1$. This will contradict our assumption that $i \in \mathscr{G}_n$.

Since $\lim_n \sum_{j\notin\mathscr{G}_n} |w_{(i,t),(j,t-l)}| = 0$ and the weight each agent places on ν_{t-l} is bounded, it is sufficient to show that

$$\sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l a_{j,t-l}$$

places non-vanishing weight on the innovation ν_{t-l} for some $l \ge 1$.

For each (j,t') such that $j \in \mathscr{G}_n$, we have

$$a_{j,t'} = \frac{\theta_{t'-1} + \sigma^{-2}s_{j,t'}}{1+\sigma^{-2}} + \epsilon_{j,t'},$$

where $\mathrm{Var}(\epsilon_{j,t'}) \to 0$. This is because

$$a_{j,t'} = \frac{(\rho^2\kappa_{i,t}^2+1)^{-1}r_{i,t} + \sigma^{-2}s_{j,t'}}{(\rho^2\kappa_{i,t}^2+1)^{-1}+\sigma^{-2}},$$

and we have $\kappa_{i,t}^2 = \mathrm{Var}(r_{i,t}-\theta_{t-1}) \to 0$.

Using this expression for $a_{j,t'}$, we obtain

$$\sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l a_{j,t-l} = \sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l\left(\frac{\theta_{t-l-1}+\sigma^{-2}s_{j,t-l}}{1+\sigma^{-2}} + \epsilon_{j,t-l}\right)$$

By the same argument as in Case 1 of the $m=1$ proof above,

$$\sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} |w_{(i,t),(j,t-l)}|$$

must be bounded (or else the contributions of signal errors to $\hat{\kappa}_i^2$ would be too large to have $i \in \mathscr{G}_n$). Therefore, it is sufficient to show that

$$\sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\rho^l\cdot\frac{\theta_{t-l-1}+\sigma^{-2}s_{j,t-l}}{1+\sigma^{-2}}$$

places non-vanishing weight on the innovation ν_{t-l} for some $l\ge 1$.

This holds for the largest l such that $\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}$ is non-vanishing. Such an l must exist, because

$$\sum_{1\le l\le m}\sum_{j\in\mathscr{G}_n} w_{(i,t),(j,t-l)}\to\frac{1}{1+\sigma^{-2}}$$

since $i\in\mathscr{G}_n$.

F.5. **Proof of Proposition 3.** For each agent i, we can write

$$a_{i,t}=w_i^s s_{i,t}+\sum_j W_{ij}\rho a_{j,t-1}=w_i^s s_{i,t}+\sum_j W_{ij}(\rho w_j^s s_{j,t}+\sum_{j'}W_{jj'}\rho a_{j',t-2}).$$

Because we assume $w_i^s<\overline{w}<1$ and$w_j^s<\overline{w}<1$ for all j, the total weight $\sum_{j,j'}W_{ij}W_{jj'}\rho$ on terms $a_{j',t-2}$ is bounded away from zero. Because the error variance of each of these terms is greater than 1, this implies agent i fails to achieve the ϵ-aggregation benchmark for $\epsilon>0$ sufficiently small.

F.6. **Proof of Proposition 4.** We prove the following statement, which includes the proposition as special cases.

Proposition 5. *Suppose the network G is strongly connected. Consider weights $\boldsymbol{W}$ and $\boldsymbol{w}^s$ and suppose they are all positive, with an associated steady state $\boldsymbol{V}_t$. Suppose either*

(1) there is an agent i whose weights are a Bayesian best response to $\boldsymbol{V}_t$, and some agent observes that agent and at least one other neighbor; or

(2) there is an agent whose weights are a naive best response to $\boldsymbol{V}_t$, and who observes multiple neighbors.

Then the steady state $\boldsymbol{V}_t$ is Pareto-dominated by another steady state.

We provide the proof in the case $m=1$ to simplify notation. The argument carries through with arbitrary finite memory.

Case (1): Consider an agent l who places positive weight on a rational agent k and positive weight on at least one other agent. Define weights $\overline{W}$ by $\overline{W}_{ij}=W_{ij}$ and $\overline{w}_i^s=w_i^s$ for all $i\ne k$, $\overline{W}_{kj}=(1-\epsilon)W_{kj}$ for all $j\le n$, and $\overline{w}_k^s=(1-\epsilon)w_k^s+\epsilon$, where W_{ij} and w_i^s are the weights at the initial steady state. In words, agent k places weight $(1-\epsilon)$ on her equilibrium strategy and extra weight ϵ on her private signal. All other players use the same weights as at the steady state.

Suppose we are at the initial steady state until time t, but in period t and all subsequent periods agents instead use weights $\overline{W}$. These weights give an alternate updating function $\overline{\Phi}$ on the space of covariance matrices. Because the weights $\overline{W}$ are positive and fixed, all coordinates of $\overline{\Phi}$ are increasing, linear functions of all previous period variances and covariances. Explicitly, the diagonal terms are

$$[\overline{\Phi}(\boldsymbol{V}_t)]_{ii} = (\overline{w}_i^s)^2\sigma_i^2 + \sum_{j,j' \leq n} \overline{W}_{ij}\overline{W}_{ij'}V_{jj',t}$$

and the off-diagonal terms are

$$[\overline{\Phi}(\boldsymbol{V}_t)]_{ii'} = \sum_{j,j' \leq n} \overline{W}_{ij}\overline{W}_{i'j'}V_{jj,t'}.$$

So it is sufficient to show the variances $\overline{\Phi}^h(\boldsymbol{V}_t)$ after applying $\overline{\Phi}$ for h periods Pareto dominate the variances in $\boldsymbol{V}_t$ for some h.

In period t, the change in weights decreases the covariance $V_{jk,t}$ of k and some other agent j, who l also observes, by $f(\epsilon)$ of order $\Theta(\epsilon)$. By the envelope theorem, the change in weights only increases the variance V_{kk} by $O(\epsilon^2)$. Taking ϵ sufficiently small, we can ignore $O(\epsilon^2)$ terms.

There exists a constant $\delta > 0$ such that all initial weights on observed neighbors are at least δ. Then each coordinate $[\Phi(\boldsymbol{V})]_{ii}$ is linear with coefficient at least δ^2 on each variance or covariance of agents observed by i.

Because agent l observes k and another agent, agent l's variance will decrease below its equilibrium level by at least $\delta^2 f(\epsilon)$ in period $t+1$. Because $\overline{\Phi}$ is increasing in all entries and we are only decreasing covariances, agent l's variance will also decrease below its initial level by at least $\delta^2 f(\epsilon)$ in all periods $t' > t+1$.

Because the network is strongly connected and finite, the network has a diameter. After $d+1$ periods, the variances of all agents have decreased by at least $\delta^{2d+2}f(\epsilon)$ from their initial levels. This gives a Pareto improvement.

Case (2): Consider a naive agent k who observes at least two neighbors. We can write agent k's period t action as

$$a_{k,t} = w_k^s s_{i,t} + \sum_{j \in N_i} W_{kj}\rho a_{j,t-1}.$$

Define new weights $\overline{W}$ as in the proof of case (1). Because agent k is naive and the summation $\sum_{j \in N_i} W_{kj}\rho a_{j,t-1}$ has at least two terms, she believes the variance of this summation is smaller than its true value. So marginally increasing the weight on $s_{k,t}$ and decreasing the weight on this summation decreases her action variance. This deviation also decreases her covariance with any other agent. The remainder of the proof proceeds as in case (1).

Appendix G. Naive Agents (online appendix)

In this section we provide rigorous detail for the analysis given in 5.1. We will describe outcomes with two signal types, σ_A^2 and σ_B^2.[52] We use the same random network model as in Section 4.2 and assume each network type contains equal shares of agents with each signal type.

We can define variances

$$V_A^\infty = \frac{\rho^2\kappa_t^2 + 1 + \sigma_A^{-2}}{\left(1+\sigma_A^{-2}\right)^2}, \qquad V_B^\infty = \frac{\rho^2\kappa_t^2 + 1 + \sigma_B^{-2}}{\left(1+\sigma_B^{-2}\right)^2} \tag{G.1}$$

where

$$\kappa_t^{-2} = 1 - \frac{1}{(\sigma_A^{-2}+\sigma_B^{-2})}\left(\frac{\sigma_A^{-2}}{1+\sigma_A^{-2}} + \frac{\sigma_B^{-2}}{1+\sigma_B^{-2}}\right).$$

Naive agents' equilibrium variances converge to these values.

Proposition 6. *Under the assumptions in this subsection:*

(1) There is a unique equilibrium on G_n.

(2) Given any $\delta > 0$, asymptotically almost surely all agents' equilibrium variances are within δ of V_A^∞ and V_B^∞.

(3) There exists $\varepsilon > 0$ such that asymptotically almost surely the ε-aggregation benchmark is not achieved, and when $\sigma_A^2 = \sigma_B^2$ asymptotically almost surely all agents' variances are larger than V^∞.

Aggregating information well requires a sophisticated response to the correlations in observed actions. Because naive agents completely ignore these correlations, their learning outcomes are poor. In particular their variances are larger than at the equilibria we discussed in the Bayesian case, even when that equilibrium is inefficient ($\sigma_A^2 = \sigma_B^2$).

When signal qualities are homogeneous ($\sigma_A^2 = \sigma_B^2$), we obtain the same limit on any network with enough observations. That is, on any sequence $(G_n)_{n=1}^\infty$ of (deterministic) networks with the minimum degree diverging to ∞ and any sequence of equilibria, the equilibrium action variances of all agents converge to V_A^∞.

G.1. **Proof of Proposition 6.** We first check that there is a unique naive equilibrium. As in the Bayesian case, covariances are updated according to equations (3.3):

[52]The general case, with many signal types, is similar.

$$\boldsymbol{V}_{ii,t} = (w^s_{i,t})^2\sigma_i^2 + \sum W_{ik,t}W_{ik',t}(\rho^2\boldsymbol{V}_{kk',t-1}+1) \text{ and } \boldsymbol{V}_{ij,t} = \sum W_{ik,t}W_{i'k',t}(\rho^2\boldsymbol{V}_{kk',t-1}+1).$$

The weights $W_{ik,t}$ and $w^s_{i,t}$ are now all positive constants that do not depend on $\boldsymbol{V}_{t-1}$. So differentiating this formula, we find that all partial derivatives are bounded above by $1 - w^s_{i,t} < 1$. So the updating map (which we call Φ^{naive}) is a contraction in the sup norm on $\mathcal{V}$. In particular, there is at most one equilibrium.

The remainder of the proof characterizes the variances of agents at this equilibrium. We first construct a candidate equilibrium with variances converging to V_A^∞ and V_B^∞, and then we show that for n sufficiently large, there exists an equilibrium nearby in $\mathcal{V}$.

To construct the candidate equilibrium, suppose that each agent observes the same number of neighbors of each signal type. Then there exists an equilibrium $\widehat{\boldsymbol{V}}^{sym}$ where covariances depend only on signal types, i.e., $\widehat{\boldsymbol{V}}^{sym}$ is invariant under permutations of indices that do not change signal types. We now show variances of the two signal types at this equilibrium converge to V_A^∞ and V_B^∞.

To estimate θ_{t-1}, a naive agent combines observed actions from the previous period with weight proportional to their precisions σ_A^{-2} or σ_B^{-2}. The naive agent incorrectly believes this gives an almost perfect estimate of θ_{t-1}. So the weight on older observations vanishes as $n \to \infty$. The naive agent then combines this estimate of θ_{t-1} with her private signal, with weights converging to the weights she uses if the estimate is perfect.

Agent i observes $\frac{|N_i|}{2}$ neighbors of each signal type, so her estimate $r^{naive}_{i,t}$ of θ_{t-1} is approximately:

$$r^{naive}_{i,t} = \frac{2}{|N_i|(\sigma_A^{-2}+\sigma_B^{-2})}\left[\sigma_A^{-2}\sum_{j\in N_i,\sigma_j^2=\sigma_A^2} a_{j,t-1} + \sigma_B^{-2}\sum_{j\in N_i,\sigma_j^2=\sigma_B^2} a_{j,t-1}\right].$$

The actual variance of this estimate converges to:

$$\text{Var}(r^{naive}_{i,t} - \theta_{t-1}) = \frac{1}{(\sigma_A^{-2}+\sigma_B^{-2})}\left[\sigma_A^{-4}Cov^\infty_{AA} + \sigma_B^{-4}Cov^\infty_{BB} + 2\sigma_A^{-2}\sigma_B^{-2}Cov^\infty_{AB}\right] \tag{G.2}$$

where Cov^∞_{AA} is the covariance of two distinct agents of signal type A and Cov^∞_{BB} and Cov^∞_{AB} are defined similarly.

Since agents believe this variance is close to 1, the action of any agent with signal variance σ_A^2 is approximately:

$$a_{i,t} = \frac{r^{naive}_{i,t} + \sigma_A^{-2}s_{i,t}}{1+\sigma_A^{-2}}.$$

We can then compute the limits of the covariances of two distinct agents of various signal types to be:

$$Cov_{AA}^{\infty} = \frac{\rho^2\kappa_t^2 + 1}{\left(1+\sigma_A^{-2}\right)^2};\ Cov_{BB}^{\infty} = \frac{\rho^2\kappa_t^2 + 1}{\left(1+\sigma_B^{-2}\right)^2};\ Cov_{AB}^{\infty} = \frac{\rho^2\kappa_t^2 + 1}{\left(1+\sigma_A^{-2}\right)\left(1+\sigma_B^{-2}\right)}.$$

Plugging into G.2 we obtain

$$\kappa_t^{-2} = 1 - \frac{1}{(\sigma_A^{-2}+\sigma_B^{-2})}\left(\frac{\sigma_A^{-2}}{1+\sigma_A^{-2}} + \frac{\sigma_B^{-2}}{1+\sigma_B^{-2}}\right).$$

Using this formula, we can check that the limits of agent variances in $\widehat{\boldsymbol{V}}^{sym}$ match equations G.1.

We must check there is an equilibrium near $\widehat{\boldsymbol{V}}^{sym}$ with high probability. Let $\zeta = 1/n$. Let E be the event that for each agent i, the number of agents observed by i with private signal variance σ_A^2 is within a factor of $[1-\zeta^2, 1+\zeta^2]$ of its expected value, and similarly the number of agents observed by i with private signal variance σ_B^2 is within a factor of $[1-\zeta^2, 1+\zeta^2]$ of its expected value. This event implies that each agent observes a linear number of neighbors and observes approximately the same number of agents with each signal quality. We can show as in the proof of Theorem 1 that for n sufficiently large, the event E occurs with probability at least $1-\zeta$. We condition on E for the remainder of the proof.

Let $\mathcal{V}_\varepsilon$ be the ε-ball around in $\widehat{\boldsymbol{V}}^{sym}$ the sup norm. We claim that for n sufficiently large, the updating map preserves this ball: $\Phi^{naive}(\mathcal{V}_\varepsilon) \subset \mathcal{V}_\varepsilon$. We have $\Phi^{naive}(\widehat{\boldsymbol{V}}^{sym}) = \widehat{\boldsymbol{V}}^{sym}$ up to terms of $O(1/n)$. As we showed in the first paragraph of this proof, the partial derivatives of Φ^{naive} are bounded above by a constant less than one. For n large enough, these facts imply $\Phi^{naive}(\mathcal{V}_\varepsilon) \subset \mathcal{V}_\varepsilon$. We conclude there is an equilibrium in $\mathcal{V}_\varepsilon$ by the Brouwer fixed point theorem.

Finally, we compare the equilibrium variances to the ϵ-aggregation benchmark and to V^{∞}. It is easy to see these variances are worse than the ϵ-aggregation benchmark for n large for some $\epsilon > 0$, and therefore by Theorem 1 also asymptotically worse than the Bayesian case when $\sigma_A^2 \neq \sigma_B^2$.

In the case $\sigma_A^2 = \sigma_B^2$, it is sufficient to show that Bayesian agents place more weight on their private signals (since asymptotically action error comes from past changes in the state and not signal errors). Call the private signal variance σ^2. For Bayesian agents, we showed in Theorem 1 that the weight on the private signal is equal to $\frac{\sigma^{-2}}{\sigma^{-2}+(\rho^2 Cov^{\infty}+1)^{-1}}$ where Cov^{∞} solves

$$Cov^{\infty} = \frac{(\rho^2 Cov^{\infty}+1)^{-1}}{[\sigma^{-2}+(\rho^2 Cov^{\infty}+1)^{-1}]^2}.$$

For naive agents, the weight on the private signal is equal to $\frac{\sigma^{-2}}{\sigma^{-2}+1}$, which is smaller since $Cov^{\infty} > 0$.

Appendix H. Socially optimal learning outcomes with non-diverse signals (online appendix)

In this section, we show that a social planner can achieve vanishing aggregation errors even when signals are non-diverse. Thus, slower rate of learning at equilibrium with non-diverse signals is a consequence of individual incentives rather than a necessary feature of the environment.

Let G_n be the complete network with n agents. Suppose that $\sigma_i^2 = \sigma^2$ for all i and $m = 1$.

Proposition 7. *Let $\varepsilon > 0$. Under the assumptions in this section, for n sufficiently large there exist weights weights $\boldsymbol{W}$ and $\boldsymbol{w}^s$ such that at the corresponding steady state on G_n, the ε-aggregation benchmark is achieved.*

Proof. An agent with a social signal equal to θ_{t-1} would place weight $\frac{\sigma^{-2}}{\sigma^{-2}+1}$ on her private signal and weight $\frac{1}{\sigma^{-2}+1}$ on her social signal. Let $w_A^s = \frac{\sigma^{-2}}{\sigma^{-2}+1} + \delta$ and $w_B^s = \frac{\sigma^{-2}}{\sigma^{-2}+1} - \delta$, where we will take $\delta > 0$ to be small.

Assume that the first $\lfloor n/2 \rfloor$ agents place weight w_A^s on their private signals and weight $1 - w_A^s$ on a common social signal r_t we will define, while the remaining agents place weight w_B^s on their private signals and weight $1 - w_B^s$ on the social signal r_t. As in the proof of Theorem 2,

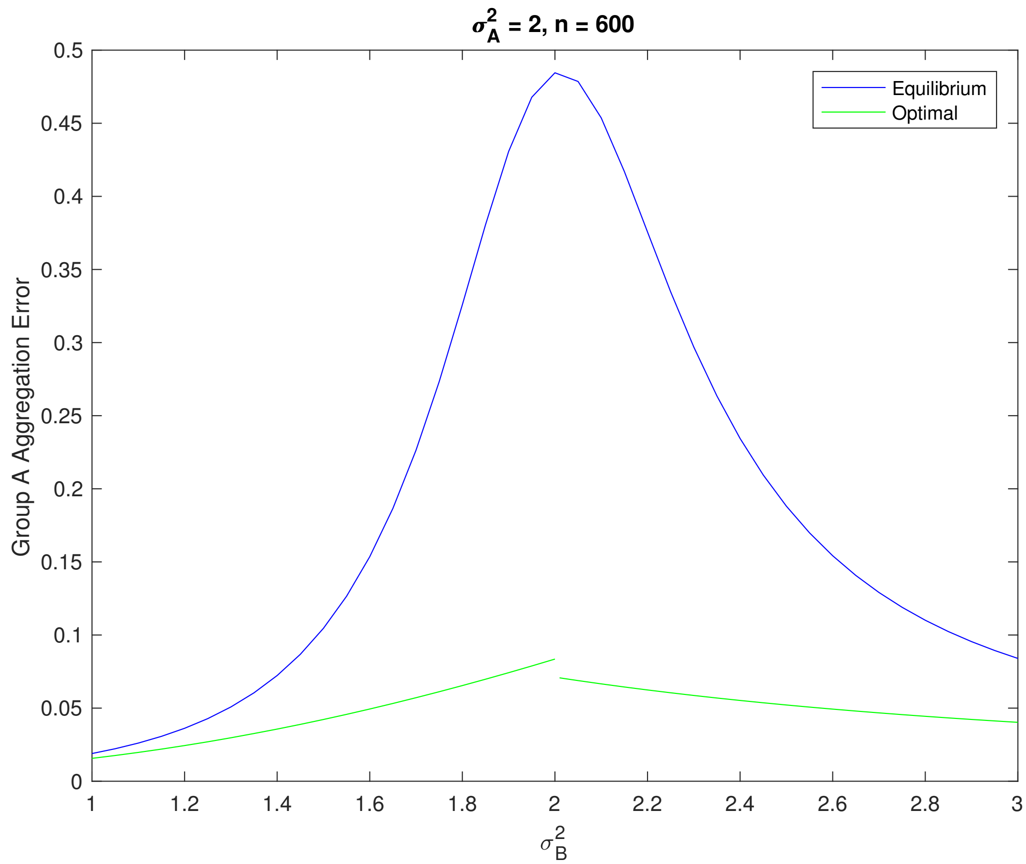
$$\frac{1}{\lfloor n/2 \rfloor} \sum_{j=1}^{\lfloor n/2 \rfloor} a_{j,t-1} = w_A^s \theta_{t-1} + (1 - w_A^s) r_{t-1} + O(n^{-1/2}),$$

$$\frac{1}{\lceil n/2 \rceil} \sum_{j=\lfloor n/2 \rfloor + 1}^{n} a_{j,t-1} = w_B^s \theta_{t-1} + (1 - w_B^s) r_{t-1} + O(n^{-1/2}).$$

There is a linear combination of these summations equal to $\theta_{t-1} + O(n^{-1/2})$, and we can take r_t equal to this linear combination. Taking δ sufficiently small and then n sufficiently large, we find that ε-perfect aggregation is achieved. □

In Figure H.1, we consider equilibrium and socially optimal outcomes with $n = 600$. Half of agents are in group A, with signal variance $\sigma_A^2 = 2$, while the other half are in group B, with signal variance σ_B^2 changing. In blue we plot average equilibrium aggregation errors for group A. In green we plot the average aggregation errors of group A when a social

Figure H.1. Social Planner and Bayesian Learning



planner minimizes the total action variance (of both groups). The weights that each agent puts on her own private signal and the other agents are set to depend only on the groups. Under these socially optimal weights agents learn very well, and heterogeneity in signal variances only has a small impact.