Online Appendix: Financial Networks and Contagion

Matthew Elliott* Benjamin Golub† Matthew O. Jackson‡

August 29, 2014

1 More on Cross-Holdings Matrices and the Induced Dependencies

In this section, we present a three-organization example to illustrate more about how the $\mathbf{A}$ and the $\mathbf{C}$ matrices can differ.

Recall our simple example from Section IG (see Figure 1). There are two organizations, $i = 1, 2$, each of which has a 50% stake in the other organization. The associated cross-holdings matrix $\mathbf{C}$ and the dependency matrix $\mathbf{A}$ are as follows. (Recall that $\widehat{C}_{ii}$ is equal to 1 minus the sum of the entries in column i of $\mathbf{C}$.)

$$\mathbf{C} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \qquad \widehat{\mathbf{C}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

A slightly richer example of potential differences between the cross-holdings and induced dependencies is as follows, with three organizations.

$$\mathbf{C} = \begin{pmatrix} 0 & 0.75 & 0.75 \\ 0.85 & 0 & 0.10 \\ 0.10 & 0 & 0 \end{pmatrix} \qquad \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} = \begin{pmatrix} 0.18 & 0.13 & 0.15 \\ 0.77 & 0.83 & 0.66 \\ 0.05 & 0.04 & 0.19 \end{pmatrix}$$

The weighted graphs of the matrix $\mathbf{C} + \widehat{\mathbf{C}}$ and the associated $\mathbf{A}$ are shown in Figure 2, illustrating the substantial differences.

*HSS, Caltech. Email: melliott@caltech.edu, web site: http://www.hss.caltech.edu/∼melliott/
†Department of Economics, Harvard University. Email: ben.golub@gmail.com, web site: http://fas.harvard.edu/∼bgolub
‡Department of Economics, Stanford University, the Santa Fe Institute, and CIFAR. Email: jacksonm@stanford.edu, web site: http://www.stanford.edu/∼jacksonm.

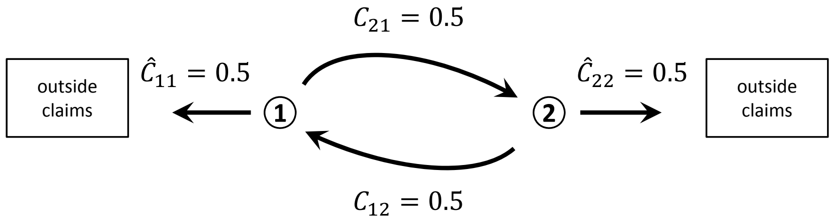


Figure 1: An illustration of cross-holdings in the two-organization example. The arrows indicate how a dollar of income arriving at one of the organizations is allocated between its direct holders and other organizations. Dollars that stay within the system are further split up. The **A** matrix describes how they are ultimately allocated.

(a) Weighted graph of $\mathbf{C} + \widehat{\mathbf{C}}$.

(b) Weighted graph of $\mathbf{A}$.

Figure 2: The widths of the edges are proportional to the sizes of cross-holdings; the arrows point in the direction of the flow of assets: from the organization that is held and to the holder. Edges pointing outside the graph in (a) reflect the private (outside) shareholders' holdings. The cross-holdings and outside holdings measured by $\mathbf{C} + \widehat{\mathbf{C}}$ can be very different from the dependency matrix $\mathbf{A}$, which measures how each organization's market value ultimately depends on the assets held by each organization.

First, note that organization 1 is almost a holding company: It is mostly owned by other organizations, and so the second two entries of the first row of $\mathbf{A}$ are much smaller than the corresponding entries in $\mathbf{C} + \widehat{\mathbf{C}}$, indicating that not much of the value of organization 1's direct holdings accrue to its private shareholders.

Also, we see that the outside shareholders of organization 2 ultimately (i.e., according to the $\mathbf{A}$ matrix) claim 66% of organization 3's direct asset holdings, even though organization 2 has only 10% of the shares of organization 3 in cross-holdings (per the $\mathbf{C}$ matrix). Intuitively, as organization 2 cross-holds 85% of organization 1, it follows that organization 2's outside shareholders indirectly have claims to organization 1's large direct stakes in both organization 2 and organization 3.

2 Debt and Other Liabilities

Throughout the paper we suppose that organizations' values depend linearly on the organizations they have holdings in, with positive slope coefficients. Debt contracts do not induce this functional form in the domain where organizations can meet the face values of their obligations. But, as we emphasize in Section IE, our analysis is centered on situations in which organizations cannot meet the face values of their obligations and must ration their counterparties. In this region, our linear model of dependencies approximates cross-holding of debt.[1]

Also, we emphasize that the discontinuous failure costs need not be triggered at the point where the value of an organization first falls below the face value of debt. There can be some regime of orderly write-down until a threshold where there is a disruption in the ability of the organization to operate, below which its value is reduced discontinuously, entering a regime of disorderly default. This is illustrated in Figure 3.

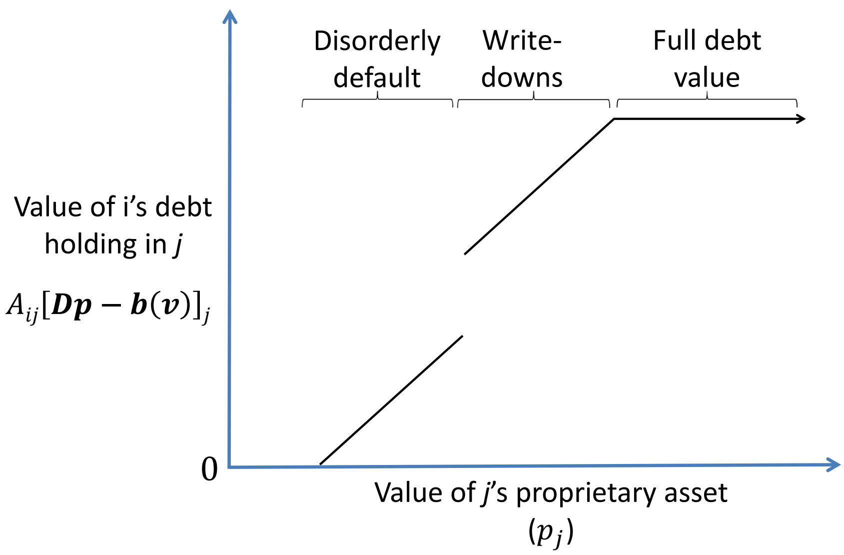


Figure 3: As the value of organization j's proprietary asset p_j decreases, a first threshold is reached at which the organization cannot pay the full face value of its liabilities. We focus on values of p_j below this threshold. After the threshold comes a region of orderly default, in which each debt holder i absorbs write-downs on the value of j's debt. As the remaining value is rationed, i's value decreases linearly in p_j until a second threshold is crossed, which we refer to as j's failure threshold. This can be interpreted as the point at which j's assets are (partially) liquidated. The resulting failure costs cause a discontinuous decrease in the value of debt holdings in j.

[1] It is not essential that all organizations be in the linear regime. If there are organizations that are "safe" and are able to pay the face value of their debts, one can model claims on them as claims on just another fundamental asset. And obligations that an organization j in the write-downs regime has to a "safe" organization can be viewed as j's obligation to an outside shareholder. In other words, since reductions in value do not feed through safe organizations, those organizations can be treated as exogenous or external to the network.

More generally, the model is easily adapted to other sorts of liabilities in addition to the linear cross-holdings that we have mainly been discussing. These could include any sort of contractual agreements, including ones contingent on the market value of the organizations (for instance, real debt commitments cannot exceed the organization's market value if there is limited liability). The basic strategy is to modify the equations for V_i to incorporate how the agreements contribute to organizations' book values (taking care to subtract liabilities as needed, so that book values do not become arbitrarily inflated). Then the fixed point of the book value equations can be computed, and the effects of various shocks studied in such a richer system.

3 Endogenously High Failure Costs and Thresholds due to Moral Hazard

Whether an organization fails depends on its failure threshold. The impact that its failure has on other organizations depends on its failure costs. If organizations have some control over their failure thresholds and costs, then we might hope that they would choose to lower them, reducing both the likelihood and the costs of failure. We show in this section that, on the contrary, organizations can actually have incentives to increase both their failure costs and thresholds.

3.1 Organization Values Can Be Endogenous

Our previous analysis has assumed that exchanges of cross-holdings or assets between organizations occur through fair trades at the current asset prices (recall Section IIIA). That was useful for illustrating the workings of the model and identifying effects of diversification and integration. However, the value to an organization of a trade depends not only on the value of the bundle of assets being received, but also on the implications of the trade for ensuing failures. Solvent or liquid organizations may have incentives to bail out insolvent or illiquid ones in order to avert a contagion (as pointed out, for example, by Leitner (2005)).[2] For instance, it can be that by relinquishing some holdings (in either assets or in another organization) an organization's value actually increases! This means that we cannot value organizations based solely on their implied underlying asset holdings but also need to consider the solvency of all other organizations. Trades can be "incentive compatible" when they are not "fair" (as evaluated by pricing the traded assets at the prices $\mathbf{p}$ and neglecting failure costs).

[2]Leitner (2005) argues that incentives for interconnected organizations to bail one another out can help them provide insurance to each other when they otherwise would not be able to commit to doing so, and that this provides an efficiency benefit from financial interconnections that can be traded off against increased systemic risk.

We first illustrate the endogeneity of values through a simple example, and then explore the associated moral hazard issues.

3.2 An Example

Consider a world with two assets and two organizations. We begin with a case where asset holdings are $D_{1\cdot} = (1, 0)$, $D_{2\cdot} = (0, 1)$. Initial cross-holdings are $C_{1\cdot} = (0, 1/2)$ and $C_{2,\cdot} = (1/2, 0)$: Each organization has a one-half stake in the other (and $\widehat{C}_{ii} = 1/2$).

From equation (5) in the paper, it is easily verified that the organizations' indirect holdings of the underlying assets are given by

$$\mathbf{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

With the initial cross-holdings, organization 1 receives 2/3 of asset 1's value while organization 2 receives 1/3. The situation for asset 2 is the reverse.

Let both asset 1 and asset 2 have price $p_1 = p_2 = 10$. Thus, without any failure costs, the (market) values of the organizations would be $v_1 = v_2 = 10$.

We let $\underline{v}_1 = 0$ and $\underline{v}_2 = 11$; let organization 2's failure costs be $\beta_2 = 6$. This means that if there are no changes in cross-holdings, from (5) the values of the two organizations are 8 and 6.[3] Suppose now that organization 1 can make a transfer to organization 2. If organization 1 were to make a transfer of 1 unit to organization 2, organization 2 would not fail and the values of the two organizations would be 9 and 11. Thus by making a transfer to organization 2, organization 1 is able to increase its value from 8 to 9! Such a payment might be a direct transfer of cash, or it could be implemented through a trade in underlying assets or cross-holdings. For example, organization 1 might simply give organization 2 an increased stake in organization 1.[4] In any case, organization 1 is incentivized to "save" organization 2.[5]

Suppose we now extend the above example to permit organization 2 to have some control over its failure costs (β_2) and failure threshold ($\underline{v}_2$). For simplicity, we suppose that organization 2 can choose from $\beta_2 \in \{0, 5, 10\}$ and from $\underline{v}_2 \in \{10, 11, 12, 13, 14\}$, and that there are no direct costs or benefits associated with the choice. Note that organization 2 can avoid failure without any intervention from organization 1 by choosing $\underline{v}_2 = 10$. However, we will see that such a choice is not in the best interests of organization 2.

[3] Values before failure costs are 10 for both organizations. Organization 2 therefore fails, and its failure cost of 6 reduces the effective value of its proprietary asset from 10 to 4. Organization 2 ultimately incurs 2/3 of this loss, while organization 1 incurs 1/3.

[4] One of the ways in which organization 1 might "save" organization 2 is to simply take over organization 2.

[5] All the parameter values in the example can be varied slightly without generating a discontinuous change in the organizations' optimal choices. In this sense the example presented is not a knife-edge case.

We assume organization 1 will "save" organization 2 if doing so weakly increases organization 1's value. If organization 2 needs saving ($\underline{v}_2 > 10$), 1's value after just saving 2 will be $v_1' = 10 - (\underline{v}_2 - 10)$, while its value will be $10 - (\beta_2/3)$ if it does not save organization 2. Organization 1 will therefore save organization 2 if and only if $\underline{v}_2 > 10$ and

$$\frac{\beta_2}{3} > (\underline{v}_2 - 10).$$

The left-hand side is the increase in value organization 1 enjoys if organization 2 remains solvent, and the right-hand side is the cost of saving organization 2—the transfer 1 must make to 2 in order for 2 to remain solvent. Table 1 below shows the transfers that organization 1 will make to organization 2 for the different values of $\underline{v}_2$ and β_2 that organization 2 can choose. These choices of $\underline{v}_2$ and β_2 then result in different values for organization 2, as shown in Table 2:

		Failure Costs β_2		
		0	5	10
	10	0	0	0
Failure	11	0	1	1
Threshold $\underline{v}_2$	12	0	0	2
	13	0	0	3
	14	0	0	0

Table 1: Transfer made from 1 to 2

		Failure Costs β_2		
		0	5	10
	10	10	10	10
Failure	11	10	11	11
Threshold $\underline{v}_2$	12	10	6 2/3	12
	13	10	6 2/3	13
	14	10	6 2/3	3 1/3

Table 2: Value of 2 after the transfer

As can be seen in Tables 1 and 2, for a fixed failure threshold, organization 2 is saved only when its failure costs are sufficiently large. Conditional on being saved, 2's value is increasing in its failure threshold; conditional on not being saved, organization 2's value is weakly decreasing in its failure threshold. For sufficiently high failure thresholds, organization 2 is never saved. And for sufficiently low failure thresholds, organization 2 doesn't fail. To maximize its utility after a bailout in this example, organization 2 must set the highest failure costs it can, and then carefully choose its failure threshold so that organization 1 is just incentivized to save it. As the table demonstrates, this requires organization 2 choosing a failure threshold of 13 and failure costs of 10.

Of course, if organizations can commit not to bail each other out, then these moral hazard problems can be avoided. However, firms have a fiduciary obligation to maximize shareholder value, even if this involves bailing out a failing organization they have a stake in. This can make it difficult for organizations to commit not to bail out one another. And absent a no-bailouts commitment device, organizations can have strong incentives to increase their failure costs and manipulate their failure thresholds.

The moral hazard problem in this example occurs absent any intervention by the government. Failure costs alone are sufficient for a moral hazard problem to arise.[6] It arises

[6] This moral hazard problem also distorts organizations' investment decisions, in terms of both their investments in risky projects and their investments in cross-holdings.

because organizations do not fully bear their failure costs. As other organizations pay some of organization i's failure costs (β_i) through the devaluation of their holdings in i, these other organizations will be prepared to expend resources bailing out i. As the proportion of i's failure costs that i pays is given by A_{ii}, a natural measure of the severity of the moral hazard problem is $1 - A_{ii}$. When $1 - A_{ii} = 0$, there is no moral hazard problem. Moreover, the extent of the moral hazard problem is monotonic in $1 - A_{ii}$ in the following sense: If $1 - A_{ii}$ is increased by redistributing shares of i from outside shareholders to other organizations, such that all other organizations' claims on i weakly increase, any organization that previously would have bailed out i faces weakly stronger incentives to bail out i, while organizations who previously would not have found it profitable to bail out i may now find it profitable to do so.

We saw in Section IIA that cascades of failure can occur, amplifying and propagating shocks if failure costs are sufficiently large and failure thresholds are sufficiently high. The analysis in this section has identified an endogenous mechanism through which organizations are willing to invest in increasing their failure costs and possibly their failure thresholds. Although such investments are valuable to an organization only in the event that it is bailed out, and in an uncertain world such bailouts may or may not be forthcoming, the misalignment of incentives due to the moral hazard problem can nevertheless result in systems endogenously conducive to cascades of failure.

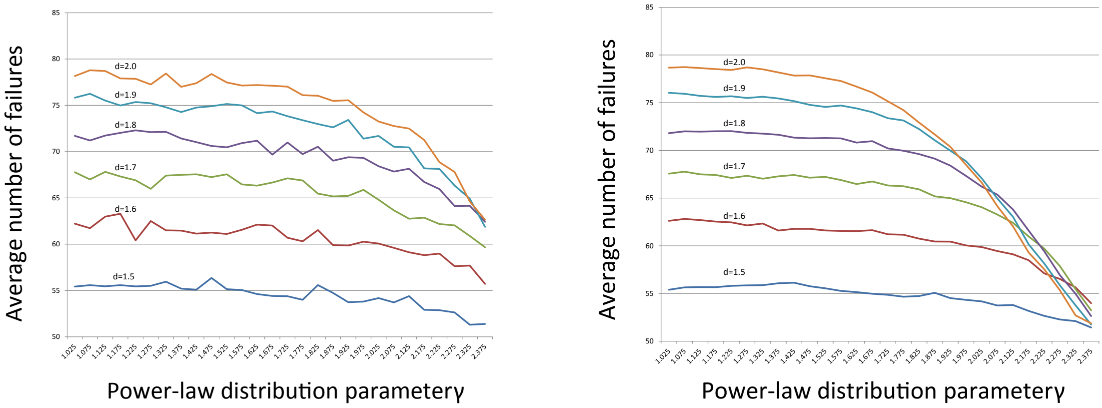
4 Additional Simulations: Alternative Degree Distributions and Correlations in Holdings

In this section we describe some additional simulations, exploring alterations of the basic model that complement the simulations of Section IV.

4.1 Power-Law Distributions

First we let the out-degree distribution for the organizations follow a (truncated) power law instead of modeling Erdos-Renyi random graphs, as in our earlier simulations. Specifically we let the outdegree d_{out} of each organization be drawn independently from a distribution $p(d_{\text{out}}) = a \cdot d_{\text{out}}^{-\gamma}$, where γ is the power-law parameter and a is a normalizing constant that ensures $p(d_{\text{out}})$ is a probability distribution. If according to a draw from this power-law distribution, organization i has an out-degree of 6, we randomly give six other organizations a $c/6$ share of i.

The objective of these simulations is to study the effect of the parameter γ on the number of failures. However, to prevent the effect of γ being conflated with changes to the expected degree d, we hold the expected degree constant by truncating the degree distribution. In other words, we pick a maximum possible degree and adjust it, for each level of γ, to hold the expected degree d constant.[7]



(a) Out-degree: Average failures of 100 organizations with out-degrees drawn from a power-law distribution.

(b) In-degree: Average failures of 100 organizations with in-degrees drawn from a power-law distribution.

Figure 4: How the average number of failures changes with the power-law parameter γ for different expected degrees, averaged over 10,000 simulations. The failure threshold is constant at $\theta = 0.95$, and the degree of integration is $c = 0.4$.

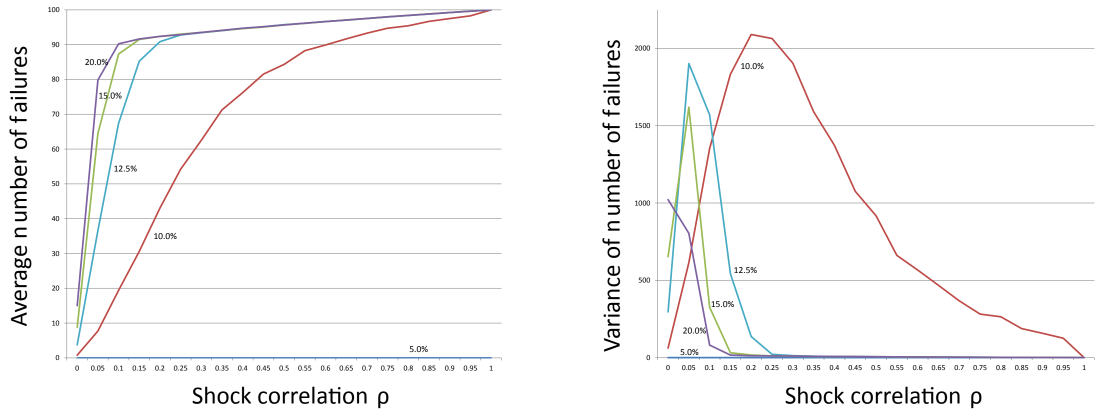
As γ increases, the number of failures decreases; there are typically larger effects as the expected degree, d, is increased even slightly. This is true both when the out-degree follows a power law and when the in-degree follows a power law.

4.2 Correlated Asset Holdings

To explore the impact of organizations' asset holdings being correlated, we run simulations where, instead of simply sending one organization's underlying asset value to zero and keeping all others at value 1, we do the following. We drop one organization's direct asset holdings by s%, and we *also* decrease some other organizations' assets by s%, where any other organization is included with a probability ρ. As ρ nears 1, all the assets drop together, whereas when ρ nears 0, only the one organization fails. As we increase ρ, we increase the number of organizations that fail together.[8]

[7]As the truncation can occur only at integer maximum degrees, we vary the maximum degree between the integer just above and just below the ideal truncation point. In all cases the normalizing constant adjusts to ensure $p(d_{\text{out}})$ is a probability distribution.

[8]This is a very simple way of introducing correlated shocks. A more detailed but nonetheless straightforward way of incorporating correlated positions would be to model holdings of many different assets that are held by multiple organizations. We could even permit people to hold negative amounts of an asset to represent shorting, although the total net position in the system must remain constant. See Section 4.3 in this appendix.



(a) Number of organizations failing by correlation of asset holdings for different initial shock magnitudes s; other parameters are $\theta = 0.95$, $c = 0.4$, $d = 3$ and $n = 100$.

(b) Variance in number of organizations failing by correlation of asset holdings for different initial shock magnitudes s; other parameters are $\theta = 0.95$, $c = 0.4$, $d = 3$ and $n = 100$.

Figure 5: How correlation in asset holdings affects the percentage of organizations failing, averaged over 5000 simulations. The horizontal axis indexes the correlation in asset holdings, measured by the proportion of organizations that suffer a shock simultaneously.

From Figures 5a and 5b, increasing the correlation of asset holdings to even a low level from a baseline of an uncorrelated system can result in relatively small shocks having highly uncertain outcomes that often result in very many failures.

4.3 Common Asset Holdings

In this section, we will start with the baseline simulation model for 100 organizations, with average degree $d = 3$ and a level of integration of $c = 0.4$, and adjust it in the following ways. First, we let each organization have holdings of two underlying assets: its own proprietary asset, and a common asset that all organizations have some (possibly negative) holdings of.[9] Each organization's holdings of the assets are determined in the following way. An organization i is selected uniformly at random and given a positive holding, x_i, in the common asset, drawn from the uniform distribution on the interval $[0, \ell]$. The parameter ℓ represents leverage, for reasons that will become clear. Next, a new organization is selected uniformly at random from the remaining organizations. This organization is the counter-party to i, and is assigned holding $-x_i$ in the common asset. This process repeats without replacement (with the uniform draws of x_i independent across these repetitions) until all organizations have been assigned a position in the common asset. By construction, the net holdings of the common asset are 0 thus far. To make the net position positive, and equal to 1, we give each organization an additional quantity $1/n$ of the common asset. At this point, the organizations will have different underlying asset values, some of which may be negative (if

[9] While our baseline model considers positive holdings, the value equations are valid provided that $\mathbf{I} - \mathbf{C}$ is invertible.

$\ell > 1/n$). To equalize the values of all organizations' underlying asset portfolios, we also give each organization a holding of a proprietary asset whose price is chosen such that the overall value of its underlying asset holdings is 1. After doing this, the sum of the values of all proprietary assets will be 99, as the net value of the common asset holdings is 1.

We then construct the cross-holdings matrix. To permit the existence of groups of organizations that are highly interconnected, we assign organizations to 10 groups and permit the probability of a link within a group to differ from the probability of a link across groups (as in the homophily simulations of Section IVB). We also correlate the assignment of organizations to these groups with their common asset holdings. To do this, we first take a weighted average of each organization's common asset holdings and an identically and independently uniformly distributed random variable on $[-\ell, \ell]$, which we call a noise term. The weight we place on common asset holdings is ρ, and the weight we place on the noise term is $1 - \rho$. Then organizations are grouped according to the decile of this weighted average. So, for $\rho = 1$, organizations are ranked according to their common asset holdings and then assigned to a group based on the decile in this ranking. When $\rho = 0$, assignment to groups is independent of common asset holdings. In this way, ρ controls the correlation between group assignments and common asset holdings.

We now form the random graph of cross-holdings among the organizations. The probability of a link within-group is weakly higher than the probability of a link across groups. These probabilities are varied while holding the expected degree constant, as in the homophily simulations. When the parameter h (standing for homophily) is 1, the probability of a link across groups is 0. More generally, the probability of a within-group link is proportional to h, while the probability of an across-group link is proportional to $1 - h$. The link probabilities are adjusted to keep the expected degree constant.

We then shock the value of the common asset and run simulations (1000 iterations for each of various combinations of the parameters ℓ, ρ, h). We look at the effect of correlating risks in a system with homophily/segregation by holding h and ℓ constant and comparing $\rho = 0$ to $\rho > 0$. We also study the effect of reducing the leverage parameter, ℓ, while holding the other parameters constant. This reduces organizations' average positions in the common asset, but also makes their exposures to the common asset more correlated (with perfect correlation when $\ell = 0$).

Interestingly, for the parameter ranges considered, adjusting the correlation of common asset positions within-group (by changing ρ) has little impact regardless of homophily. In contrast, adjusting the leverage parameter has a substantial impact. For even small shocks to the common asset of 5 percent, large cascades occur (across the range of other parameters) for $\ell > 1.5$. Note that for these higher levels of leverage, the correlation in exposure to the common asset is actually lower.

The threshold value of the parameter ℓ for which a large cascade occurs decreases in the size of the shock. However, for large shocks to the common asset of 20 percent, increasing the parameter ℓ reduces the extent of cascades. Intuitively, a sufficiently large parameter ℓ means that some organizations have significant negative positions in the common asset (short positions—e.g., Goldman Sachs in the 2008 crisis), and their value initially increases as a result of the shock. This can be sufficient for them, and those who have holdings in them, to survive the failure of many other organizations.

5 Using the Dependency Matrix

This section validates direct manipulation of the dependency matrix $\mathbf{A}$. First, Proposition OA1 below shows that absent any discontinuities (i.e., with failure costs of zero for all organizations), any change in $\mathbf{C}$ or $\mathbf{A}$ can be represented as changes in $\mathbf{D}$ alone. However, we may want to hold $\mathbf{D}$ fixed and ask when there is a $\mathbf{C}$ giving rise to a given $\mathbf{A}$. In other words, we want to have an explicit description of the image of the function $\mathbf{C} \mapsto \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ (i.e., the image under this map of all $\mathbf{C}$ satisfying our maintained assumptions). Proposition OA2 below identifies a simple necessary and sufficient condition for any given $\mathbf{A}$ to be in this image.

Proposition **OA1.** Assuming there are no failures, for any pair $\mathbf{D}, \mathbf{C}$, there is a pair $\mathbf{D}', \mathbf{C}'$, with $\mathbf{C}'$ being the matrix of 0's (and $\widehat{\mathbf{C}}'$ being the identity), that results in the same organization values for any underlying asset prices $\mathbf{p}$. Similarly, for any $\mathbf{A}, \mathbf{D}$ there is a $\mathbf{D}'$ that results in the same organization values for any underlying asset prices $\mathbf{p}$, with $\mathbf{C} = \mathbf{0}$.

Proposition OA1 follows directly from letting

$$\mathbf{D}' = (\widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1})\mathbf{D} = \mathbf{A}\mathbf{D}.$$

Thus, in the absence of failure, it is simply the indirect holdings of underlying assets that matter, and so one can work with equivalent direct holdings in studying organizations' values.

The proposition implies that instead of considering trades in cross-holdings, when we are working to understand what might trigger a *first* failure (so that no failure has yet occurred) there is always some trade in underlying assets that replicates any given trade in cross-holdings.

However, in practice, at least some of the underlying assets are not directly tradeable and so can be exchanged only through cross-holdings.[10] To work in the underlying asset space, we therefore want to know when trades of underlying assets can be replicated through an exchange of cross-holdings, keeping the organizations' asset holdings ($\mathbf{D}$) constant. Proposition OA2 answers this question. We say a square matrix $\mathbf{C}$ is *permissible* if the diagonal is zero, there are no negative entries, and each column sums to strictly less than 1; the diagonal matrix $\widehat{\mathbf{C}}$ is obtained from any permissible $\mathbf{C}$ by letting $\widehat{C}_{jj} = 1 - \sum_{i \neq j} C_{ij}$.

[10]If all underlying assets were freely tradeable, then there would be no reason for any cross-holdings. Any portfolio of claims to underlying assets held through cross-holdings could be replicated as direct holdings and without any risk of devaluation through failure.

PROPOSITION **OA2.** There is a permissible $\mathbf{C}$ such that $\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ if and only if $\mathbf{A}$ is invertible and column-stochastic, and the following conditions hold: $(\mathbf{A}^{-1})_{ii} > 0$ for all i and $(\mathbf{A}^{-1})_{ij} \leq 0$ whenever $j \neq i$.

Proof of Proposition OA2: First, some preliminaries. Recall from (5) in the main text that

$$\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}.$$

If $\mathbf{A}$ is invertible, manipulating this equation yields the following string of equivalences:

$$\begin{aligned} \mathbf{A}^{-1} &= (\widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1})^{-1} \\ \mathbf{A}^{-1} &= (\mathbf{I} - \mathbf{C})\widehat{\mathbf{C}}^{-1} \\ \mathbf{A}^{-1}\widehat{\mathbf{C}} &= \mathbf{I} - \mathbf{C} \\ \mathbf{C} &= \mathbf{I} - \mathbf{A}^{-1}\widehat{\mathbf{C}}. \end{aligned} \tag{OA-1}$$

Considering entry (i, i) of this matrix equation, and recalling that $\widehat{\mathbf{C}}$ is a diagonal matrix:

$$C_{ii} = 1 - (\mathbf{A}^{-1})_{ii}\widehat{C}_{ii}.$$

If $C_{ii} = 0$, the equations of (OA-1) that correspond to the diagonal entries are equivalent to the following collection of equations (as i ranges across all values):

$$\widehat{C}_{ii} = 1/(\mathbf{A}^{-1})_{ii}. \tag{OA-2}$$

This allows us to express the right-hand side of (OA-1) in terms of just $\mathbf{A}$. To summarize, we have shown:

LEMMA **OA1.** Whenever $\mathbf{A}$ is invertible and $\mathbf{C}$ has a zero diagonal, the system consisting of (OA-1) and (OA-2) is equivalent to the system

$$\begin{aligned} \mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1} &\quad \text{and} \\ \widehat{C}_{jj} = 1 - \sum_{i \neq j} C_{ij} &\quad \text{for all } j. \end{aligned}$$

Now we can prove the "only if" direction of the proposition. Take a permissible $\mathbf{C}$. It follows directly from the formula $\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ and from the existence of the inverse on the right-hand side (which we established in the main text) that $\mathbf{A}$ is invertible. And since a permissible $\mathbf{C}$ has zero diagonal, we can apply the lemma. Using that $\widehat{\mathbf{C}}$ has strictly positive diagonal entries, it follows from (OA-2) that $(\mathbf{A}^{-1})_{ii} > 0$ for every i. And since any permissible $\mathbf{C}$ has nonnegative off-diagonal entries, and $\widehat{\mathbf{C}}$ has strictly positive diagonal entries, we deduce from (OA-1) that $(\mathbf{A}^{-1})_{ij} \leq 0$ whenever $j \neq i$. Footnote 18 in the main text shows that $\mathbf{A}$ is column-stochastic.

Next, we prove the "if" direction of the proposition. Given an invertible and column-stochastic $\mathbf{A}$, we will let $\mathbf{C}$ and $\widehat{\mathbf{C}}$ be the matrices defined by (OA-1) and (OA-2). It follows immediately from these definitions that $\mathbf{C}$ has 0's on its diagonal. Lemma OA1 then gives that $\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1}$ and that $\widehat{\mathbf{C}}$ satisfies the equation $\widehat{C}_{jj} = 1 - \sum_{i \neq j} C_{ij}$, for every j. Thus, to finish the proof, it suffices to check that $\mathbf{C}$ is permissible.

First we prove that $\mathbf{C} + \widehat{\mathbf{C}}$ has columns summing to 1. By hypothesis, $\mathbf{A}$ is column-stochastic, so $\mathbf{1}^T\mathbf{A} = \mathbf{1}^T$, where $\mathbf{1}$ is a column of 1's. Now post-multiply by $\mathbf{A}^{-1}$. We then find that $\mathbf{1}^T = \mathbf{1}^T\mathbf{A}^{-1}$ and so $\mathbf{A}^{-1}$ also has columns summing to 1. Therefore, $\sum_{i=1}^{n}(\mathbf{A}^{-1})_{ij}\widehat{C}_{jj} = \widehat{C}_{jj}\sum_{i=1}^{n}(\mathbf{A}^{-1})_{ij} = \widehat{C}_{jj}$. Adding $\widehat{\mathbf{C}}$ to both sides of equation (OA-1), we then have, for any j

$$\sum_{i=1}^{n} C_{ij} + \widehat{C}_{ij} = \sum_{i=1}^{n}\left[I_{ij} - (\mathbf{A}^{-1})_{ij}\widehat{C}_{jj} + \widehat{C}_{ij}\right] = 1 - \widehat{C}_{jj} + \widehat{C}_{jj} = 1$$

Assuming $(\mathbf{A}^{-1})_{ii} > 0$, we deduce $\widehat{C}_{ii} > 0$ from (OA-2). Combining this with (OA-1) and the assumption that $(\mathbf{A}^{-1})_{ij} \leq 0$ whenever $j \neq i$ guarantees that the off-diagonal entries of $\mathbf{C}$ are nonnegative. These observations show that $\mathbf{C}$ has nonnegative entries and that its columns sum to less than 1. And that concludes the demonstration that $\mathbf{C}$ is admissible. ∎

6 Bounds on the Dependency Matrix

We provide some useful upper bounds on the possible values of the dependency matrix $\mathbf{A}$.

Let $\bar{c} = \max_k(1 - \widehat{C}_{kk})$; let

$$\overline{A}_{ij} = \widehat{C}_{ii}\frac{\bar{c}}{1-\bar{c}}\max_{k\neq i}\frac{C_{ik}}{1-\widehat{C}_{kk}}$$

and let

$$\overline{A}_{ii} = \widehat{C}_{ii}\left(1 + \frac{\bar{c}}{1-\bar{c}}\max_{k\neq i}\frac{C_{ik}}{1-\widehat{C}_{kk}}\right).$$

Lemma **OA2.** $\overline{A}_{ij}$ is an upper bound on A_{ij} for all i and j. Therefore, if $\widehat{C}_{ii} = 1 - c$ for all i,[11] then $A_{ij} \leq \max_{k\neq i} C_{ik}$ for each i with $j \neq i$, and $A_{ii} \leq (1-c) + \max_{k\neq i} C_{ik}$.

Proof. Recall that

$$\mathbf{A} = \widehat{\mathbf{C}}(\mathbf{I} - \mathbf{C})^{-1},$$

or, equivalently, that

$$\mathbf{A} = \widehat{\mathbf{C}}\sum_{t=0}^{\infty}\mathbf{C}^t.$$

[11] So that each organization has c of its proprietary holdings shared out to other organizations and retains $1 - c$.

Let $\overline{\mathbf{C}}$ be the matrix for which we set $\overline{C}_{ij} = \frac{C_{ij}}{1-\widehat{C}_{jj}}$. Then

$$\mathbf{A} \leq \widehat{\mathbf{C}} \sum_{t=0}^{\infty} \overline{c}^t \overline{\mathbf{C}}^t.$$

Note that $\overline{\mathbf{C}}$ is a column-stochastic matrix. It follows that $\overline{\mathbf{C}}^{t-1}$ is also column-stochastic for any $t \geq 1$ (because it is a column-stochastic matrix raised to a power). Write $\overline{\mathbf{C}}^t = \overline{\mathbf{C}}\,\overline{\mathbf{C}}^{t-1}$. From this, given the fact that $\overline{\mathbf{C}}^{t-1}$ is column-stochastic for each t, it follows that the entry (i,j) of $\overline{\mathbf{C}}^t$ is no more than $\max_{k \neq i} \frac{C_{ik}}{1-\widehat{C}_{kk}}$. Also, note that for $t = 0$, entry (i,j) of $\overline{\mathbf{C}}^t$ when $j \neq i$ is 0. Thus, for $i \neq j$,

$$A_{ij} \leq \widehat{C}_{ii} \sum_{t=1}^{\infty} \overline{c}^t \max_{k \neq i} \overline{C}_{ik}.$$

Then since $1/\sum_{t=1}^{\infty} \overline{c}^t = \overline{c}/(1-\overline{c})$ it follows that

$$A_{ij} \leq \widehat{C}_{ii} \frac{\overline{c}}{1-\overline{c}} \max_{k \neq i} \overline{C}_{ik},$$

This is the claimed expression for $j \neq i$. For $j = i$ we have entry (i,i) of $\overline{\mathbf{C}}^0$ being 1, and the rest of the reasoning is the same. The simplifications when $\widehat{C}_{ii} = 1 - c$ for all i follow directly. □

7 Multiple Equilibria and Discontinuities in Organizations' Values

In the absence of any failure issues, equation (5) from the paper simplifies to $\mathbf{v} = \mathbf{A}\,[\mathbf{D}\mathbf{p}]$, which is just a standard pricing equation describing how the values of organizations depend on the primitive asset values. The novel and interesting part comes from the failure costs $\mathbf{b}(\mathbf{v})$. These terms generate several complexities that equation (5) illuminates.

The presence of failure introduces several forms of discontinuity which result in multiple equilibria. Discontinuities in the value of a given organization i can come from two sources. The basic form is that the failure costs of organization i can be triggered when the values of underlying assets fall, which can, through either direct holdings or cross-holdings, lead i to hit its failure threshold. The other form is due to another organization, in which i has cross-holdings, hitting its failure threshold, which then leads to a discontinuous drop in the value of i's holdings and consequently its value.

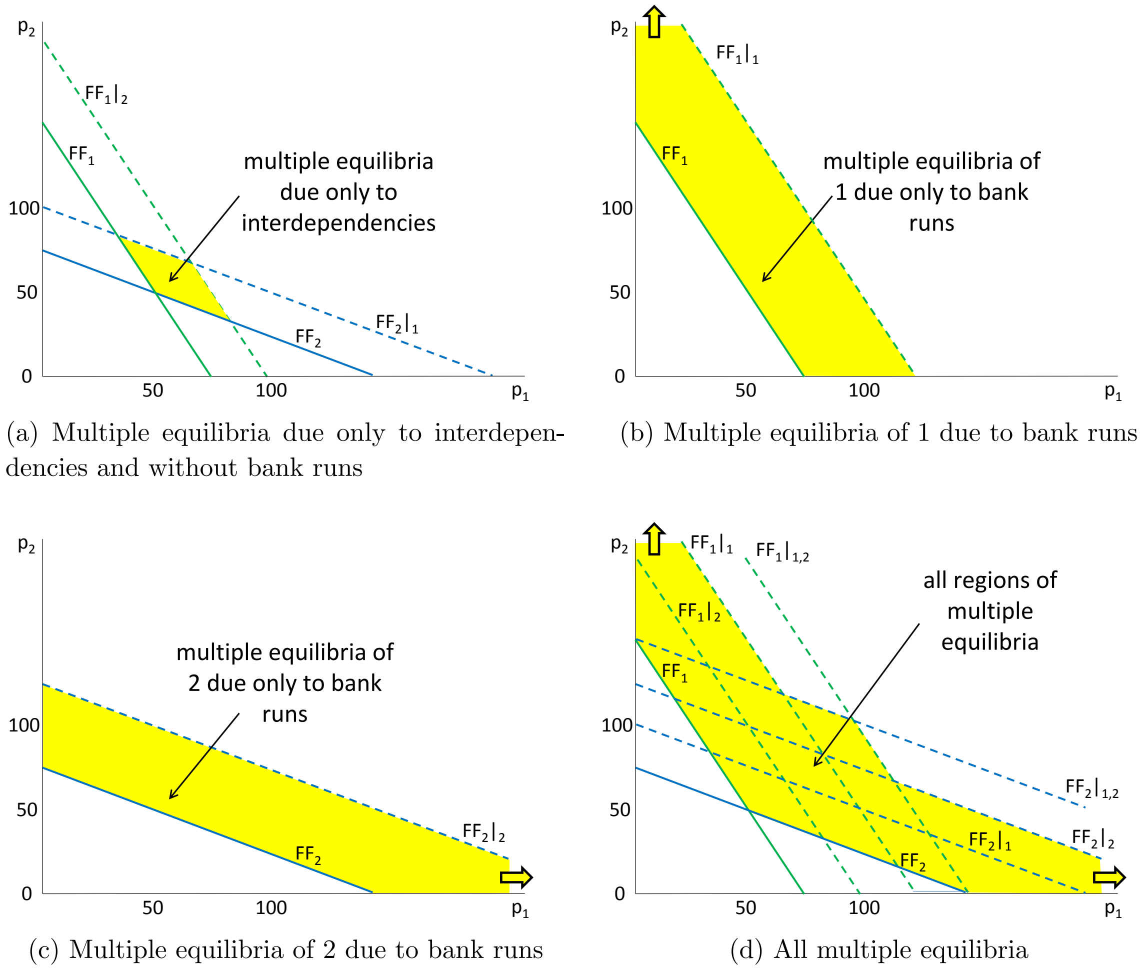
In terms of multiplicities of equilibria, there are also different ways in which these can occur. The first is that taking other organizations' values and the value of underlying assets as fixed and given, there can be multiple possible consistent values of organization i that solve equation (5). There may be a value of v_i satisfying equation (5) such that $1_{v_i<\underline{v}_i} = 0$ and another value of v_i satisfying equation (5) such that $1_{v_i<\underline{v}_i} = 1$; this can occur even when all other prices and values are held fixed. This generates a first source of multiple equilibria; this corresponds to the standard story of self-fulfilling bank runs (discussed in classic models such as that of Diamond and Dybvig (1983)).

The second way for multiplicity of equilibria to arise is through the interdependence of the values of the organizations: The value of organization i depends on the value of organization j, while the value of organization j depends on the value of organization i. And given the discontinuities possible in prices due to failure costs, there can be multiple solutions. There might then be two consistent joint values of i and j: one consistent value in which both i and j fail, and another consistent value in which both i and j remain solvent. This second source of multiple equilibria is different from the individual bank-run concept, as here organizations fail because people expect other organizations to fail, which then becomes self-fulfilling.

Although governments may be able to give assurances (e.g., by insuring deposits) that manipulate expectations regarding the self-fulfilling value of a single organization, it seems more difficult to control expectations when an organization's value depends on the expected values of many other organizations. For example, an organization's value can depend on the expected value of an organization that falls under the regulatory oversight of another government. Suppose organizations i and j have cross-holdings in each other and organization j also has cross-holdings in organization k. Investors in organization i may then become less confident that investors will keep their money in organization j, or less confident that the investors in j will have confidence in them or in the investors in k, and so on.

8 Including Multiple Equilibria Due to Bank Runs

This section extends the example in Section IIB. The same parameter values are used in Figure 6 as were used in Section IIB and Figure 1, although the scale of the axes has been adjusted. As can be seen, the scope for multiple equilibria increases a great deal once bank runs are permitted. Note that i's failure threshold conditional on i failing is shifted out twice as far as i's failure threshold conditional on j failing because i effectively pays $2/3$ of his failure costs but only $1/3$ of j's. As shown in Figure 6(d), there is a large set of prices for which it is consistent for both 1 and 2 to fail. In these outcomes, total failure costs of 100 are incurred and failure costs of 50 are paid by each organization.



(a) Multiple equilibria due only to interdependencies and without bank runs

(b) Multiple equilibria of 1 due to bank runs

(c) Multiple equilibria of 2 due to bank runs

(d) All multiple equilibria

Figure 6: The set of multiple equilibria is much larger once bank runs are permitted. Nevertheless, the interdependencies provide an additional source of multiplicity even when bank runs are permitted. The notation $\mathrm{FF}_i|_j$ refers to the frontier (in the space of underlying asset prices) separating i's solvency from insolvency, conditional on j's failure costs being subtracted from j's assets.

9 Best-Case and Worst-Case Tradeoffs

We now return to considering multiplicity of equilibria due to the interdependencies between organizations. We identify a tension between limiting failures in the best-case equilibrium and worst-case equilibrium. Trades that prevent any organizations failing in the best-case outcome can also make more organizations fail in the worst-case outcome.

We say that cross-holdings are *best-case safest* when they maximize the percentage decrease in asset prices that would be necessary for a first organization to fail. More formally, $(\mathbf{C}, \mathbf{D})$ are said to be best-case safest at prices $\mathbf{p}$ if, in the best-case equilibrium, all organizations survive and the holdings $(\mathbf{C}, \mathbf{D})$ solve the following maximization problem:

$$\max_{\mathbf{C},\mathbf{D}} \min_i \frac{v_i - \underline{v}_i}{v_i},$$

where v_i depends on $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{p}$.

It is possible for all organizations to fail only if the total value of primitive assets less all failure costs can be allocated in a way that leaves all organizations below their failure thresholds. Such an allocation exists if and only if

$$\sum_k p_k - \sum_i \beta_i < \sum_i \underline{v}_i.$$

Proposition **OA3.** Assume $\mathbf{p}$ has only positive entries; organizations' failure costs are a constant proportion γ of the value of their direct asset holdings (i.e. $\beta_i = \gamma \sum_k D_{ik} p_k$) and it is possible for all organizations to fail in some equilibrium. Then any asset holdings that are best-case safest at prices $\mathbf{p}$ also result in all organizations failing in the worst-case equilibrium at prices $\mathbf{p}$.

Proof: If no organization fails, then their market values are

$$\mathbf{v} = \mathbf{A}\mathbf{D}\mathbf{p}.$$

The *best-case safest* holdings maximize the percentage loss that any organization can suffer without failing. As all assets have positive value, this requires equalizing the proportional loss in value each organization must suffer to fail. If this were not equalized, reallocating assets at the margin from the set of organizations furthest from their failure constraints to those organizations closest to them would increase the percentage loss in value that any organization could sustain without failing. Thus, in a best-case safest asset allocation,

$$\mathbf{v} = \mathbf{A}\mathbf{D}\mathbf{p} = \theta \underline{\mathbf{v}} \tag{OA-3}$$

for some scalar θ.

As, by assumption, it is possible to allocate the combined bankruptcy cost of all organizations in a way that would cause them all to fail, we have

$$\sum_k p_k - \sum_i \beta_i < \sum_i \underline{v}_i.$$

Using the fact that $\mathbf{A}$ and $\mathbf{D}$ are column-stochastic, and that failure costs are a constant proportion γ of the value of organizations' direct asset holdings, we can rewrite the above equation as:

$$\sum_j \sum_i \sum_k (1-\gamma) A_{ij} D_{ik} p_k < \sum_i \underline{v}_i.$$

Now, using (OA-3) we rewrite the left hand side as $(1-\gamma)\theta \sum_i \underline{v}_i$ and conclude that

$$(1-\gamma)\theta < 1.$$

Now we will use this to show that it is an equilibrium for all organizations to fail. Note that, if we take all organizations' bankruptcy costs out of their values, we have:

$$\mathbf{A}(\mathbf{Dp} - \boldsymbol{\beta}) = \mathbf{ADp}(1-\gamma) = (1-\gamma)\theta\underline{\mathbf{v}} < \underline{\mathbf{v}}$$

Thus, all organizations are below their failure thresholds, and therefore it is an equilibrium for all organizations to fail. ∎

Proposition OA3 illustrates that if trades are undertaken with the sole purpose of achieving the best-case safest outcome, these same trades can also result in the worst possible outcome occurring in the worst-case equilibrium—all organizations failing.

10 Details on Cascades of Default in Europe

We first discuss the data used and then provide the calculations for $\underline{v}_i$, the failure thresholds. There are data available from the Bank of International Settlements on aggregated cross-liabilities between countries on both an immediate-borrower basis (which reports all contracts) and a final-borrower basis (which nets out contracts with intermediaries, replacing them with contracts between the final parties). If two parties trade through an intermediary, we assume that the intermediary writes separate contracts with the two parties (or acts as some kind of guarantor). In this case, default by the intermediary would affect both parties, and it is appropriate to use the intermediate-borrower basis data.[12]

The calculations of $\underline{v}_i$ are based on the peak GDPs from 2008. The normalized 2008 GDPs (relative to Portugal's GDP in 2011) are

$$\begin{pmatrix} 12.0 \\ 15.3 \\ 1.5 \\ 9.7 \\ 1.1 \\ 6.7 \end{pmatrix}.$$

This leads to 2008 values, based on the $\mathbf{A}$ matrix, of

[12] Note that calculating the $\mathbf{A}$ matrix is far more involved than just looking at the final borrower basis data.

$$\mathbf{v}_0 = \mathbf{A}\mathbf{p} = \begin{pmatrix} 0.71 & 0.13 & 0.13 & 0.17 & 0.07 & 0.11 \\ 0.18 & 0.72 & 0.12 & 0.11 & 0.09 & 0.14 \\ 0.00 & 0.00 & 0.67 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.12 & 0.03 & 0.70 & 0.03 & 0.05 \\ 0.01 & 0.00 & 0.02 & 0.00 & 0.67 & 0.02 \\ 0.03 & 0.03 & 0.02 & 0.02 & 0.14 & 0.68 \end{pmatrix} \cdot \begin{pmatrix} 12.0 \\ 15.3 \\ 1.5 \\ 9.7 \\ 1.1 \\ 6.7 \end{pmatrix} = \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 5.4 & \text{(Spain)} \end{pmatrix}.$$

Thus

$$\underline{\mathbf{v}} = \theta \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 5.4 & \text{(Spain)} \end{pmatrix}, \quad \text{and} \quad \beta = \frac{\theta}{2} \begin{pmatrix} 13.1 & \text{(France)} \\ 15.4 & \text{(Germany)} \\ 1.0 & \text{(Greece)} \\ 9.8 & \text{(Italy)} \\ 1.0 & \text{(Portugal)} \\ 5.4 & \text{(Spain)} \end{pmatrix}.$$

11 Lemmas in the Proof of Proposition 3

Here we prove Lemmas 3 and 4, which are used in the proof of Proposition 3. We maintain the notation of that proof.

Proof of Lemma 3: Let $\underline{d}_* = \max\{\underline{d}, 1\}$. By the Neumann series (equation (A1)) applied to the structure of the present random graph, we have (absent any failures)

$$\mathbf{v} = (1-c)\sum_{p=0}^{\infty} \mathbf{C}^p \mathbf{1} \leq (1-c)\sum_{p=0}^{\infty} c^p (\underline{d}_*^{-1}\mathbf{G})^p \mathbf{1} \leq (1-c)\sum_{p=0}^{\infty} \left(c\frac{\overline{d}}{\underline{d}_*}\right)^p \mathbf{1},$$

where in the first inequality we have used a bound $C_{ij} \leq G_{ij}/\underline{d}_*$ on the entries of $\mathbf{C}$, and in the second we have used the fact that $\mathbf{G}^p\mathbf{1} \leq \overline{d}^p\mathbf{1}$, which is easy to verify by induction and the fact that $\overline{d}$ is the maximum degree in the graph $\mathbf{G}$. This establishes that $v_i \leq \widetilde{v}_{\max}$ for each i; the rewriting of the summation the way we have done in the definition of $\widetilde{v}_{\max}$ is valid as long as $c\overline{d}/\underline{d}_* < 1$, which we assume in footnote 50 of the main text. The argument for the inequality $v_i \geq \widetilde{v}_{\min}$ is analogous: We use that $C_{ij} \geq G_{ij}/\overline{d}$ and then the fact that $\mathbf{G}^p\mathbf{1} \geq \underline{d}^p\mathbf{1}$. ∎

Proof of Lemma 4: Fix a j as defined in the statement. We will prove the lemma by translating it into a statement about the probability of a certain event in a suitably defined Markov chain, which turns out to be more intuitive to establish. Let $\overline{\mathbf{C}}$ be defined by $\overline{C}_{xy} = G_{xy}/d_y$. Consider a Markov process (X_t) with state space $\{0, 1, 2, \ldots, n\}$ and initial state i. The state 0 is an absorbing state. From state $x \geq 1$, with probability $1-c$, a transition occurs to state 0, and otherwise, the probability of moving to any state $y \geq 1$ is given by $\overline{C}_{yx}$. Observing that $\mathbf{C} = c \cdot \overline{\mathbf{C}}$, it is easy to verify that $Q_{ji} = \left(\sum_{p=2}^{\infty} \mathbf{C}^p\right)_{ji}$ is the probability of the following event E_j: There is some $t \geq 2$ such that $X_t = j$.

We will show that, once k is large enough, the probability of E_j is at most $\varepsilon/[\overline{d}(1-c)]$ for each j such that $G_{ji} = 1$; since there are at most $\overline{d}$ such j, we then conclude by the union bound that the probability of $\bigcup_{j:A_{ji}=1} E_j$ is at most $\varepsilon/(1-c)$. Let T be the (random) set of nodes reached with positive probability from i in exactly two steps. For a fixed constant a, let M be the (random) set of nodes with a directed path of length at most a to j. Clearly, $|M| \leq \sum_{k=0}^{a} \overline{d}^a \leq \overline{d}^{a+1}$ (recall that the maximum degree possessed by any node in $\mathbf{G}$ is $\overline{d}$). In other words, M constitutes a very small fraction of the nodes in the graph as the graph becomes large. Applying the Bollobás configuration model as outlined in Section 2.1 of Cooper and Frieze (2004) to make precise the fact that T and M are essentially independent conditional on i, we deduce that we can find n large enough so that the probability that $T \cap M$ is nonempty is at most $\varepsilon/[2\overline{d}(1-c)]$. From this we can conclude that $Q_{ji} \leq \varepsilon/[2\overline{d}(1-c)] + (1-c)^a$. The first term is an upper bound on the probability that $T \cap M$ is nonempty. On the complementary event where $T \cap M$ is empty, the following holds: To return to j via a path of length at least 2, the Markov process has to take more than a steps (by definition of T and M). At each of these steps, conditional on the history, the process has a probability $1-c$ of being absorbed at 0. Taking a large shows that $Q_{ji} \leq \varepsilon/[\overline{d}(1-c)]$.

References

Cooper, C. and A. Frieze (2004): "The Size of the Largest Strongly Connected Component of a Random Digraph with a Given Degree Sequence," *Combinatorics, Probability, and Computing*, 13, 319–37.

Diamond, D. W. and P. H. Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91(3), 401–19.

Leitner, Y. (2005): "Financial Networks: Contagion, Commitment, and Private Sector Bailouts," *Journal of Finance*, 60(6), 2925–53.