A Network Approach to Public Goods

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Cambridge

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- Characterize **efficient frontier** as well as **Lindahl outcomes** (with strategic foundations)
  - in terms of **eigenvalues** and **eigenvectors** of a matrix of marginal payoff relationships.
  - Conceptually: market outcomes $\leftrightarrow$ network centrality measures.

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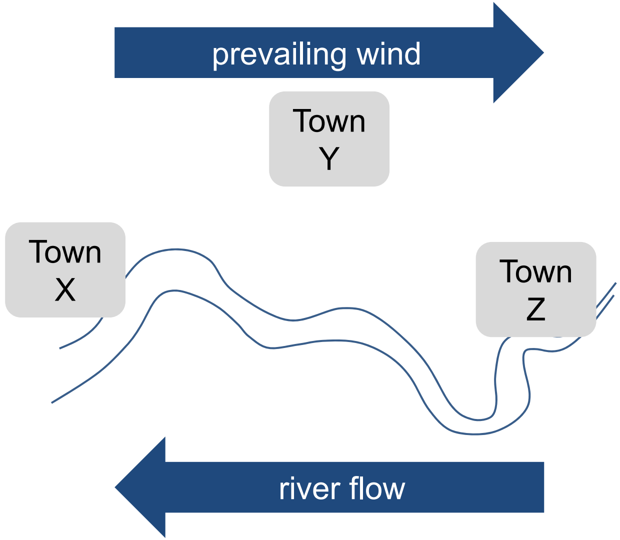
- costly actions: $\frac{\partial u_i}{\partial a_i} < 0$;
- positive externalities: $\frac{\partial u_i}{\partial a_j} \geq 0$ if $i \neq j$.

The Environment: An Example



$\boldsymbol{B}$: The (Marginal) Benefits Matrix

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$$B_{ij} = \begin{cases} \dfrac{\partial u_i/\partial a_j}{-\partial u_i/\partial a_i} & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

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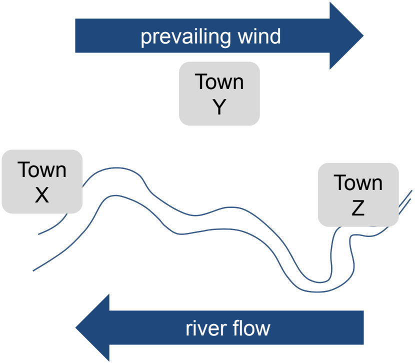
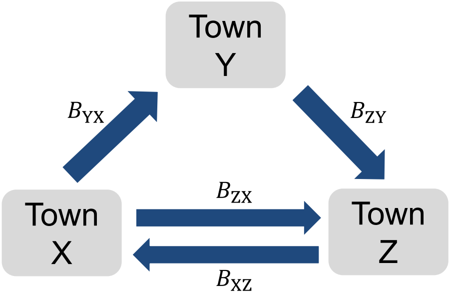
We assume $\boldsymbol{B}(\boldsymbol{a})$ is irreducible for all $\boldsymbol{a}$.

The Benefits Matrix

We can think of $\boldsymbol{B}(\boldsymbol{a})$ as a network.





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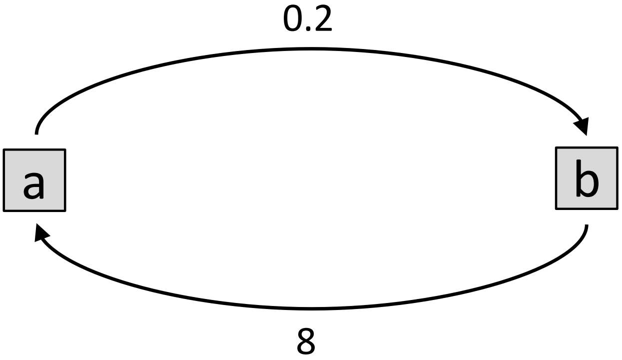
Example: Is a Pareto Improvement Possible?

$$\boldsymbol{B}(\mathbf{0}) = \begin{bmatrix} 0 & 8 \\ 0.2 & 0 \end{bmatrix}$$



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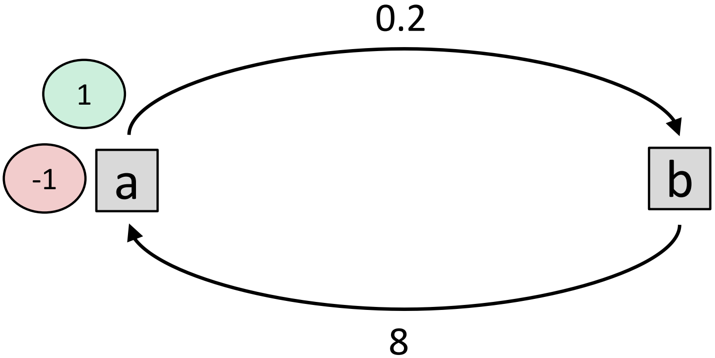

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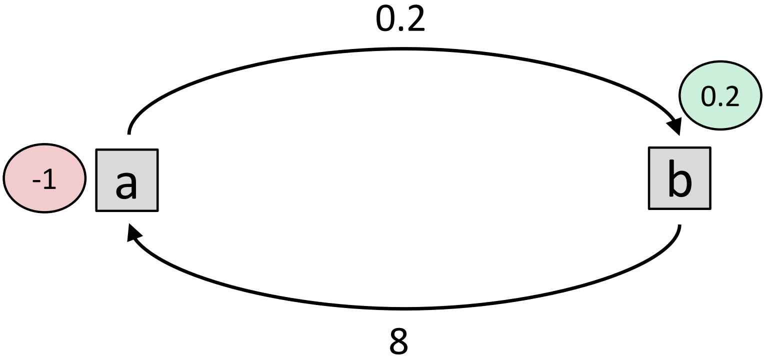

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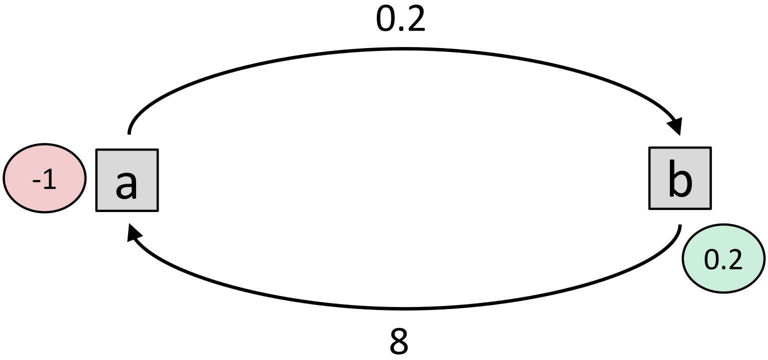

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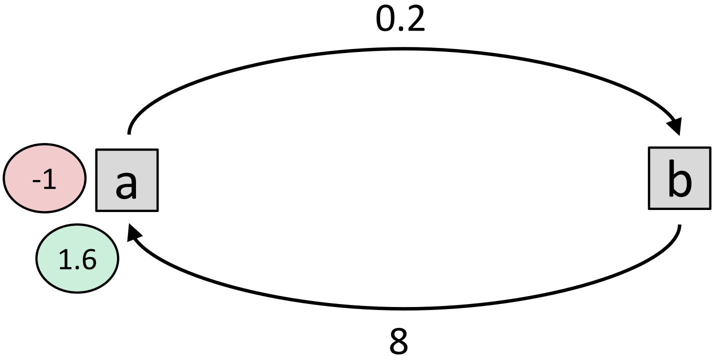

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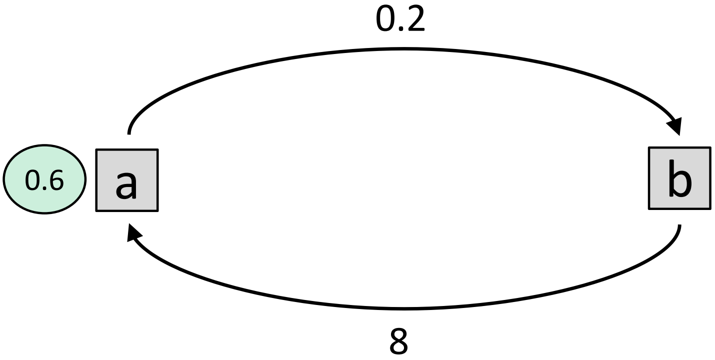


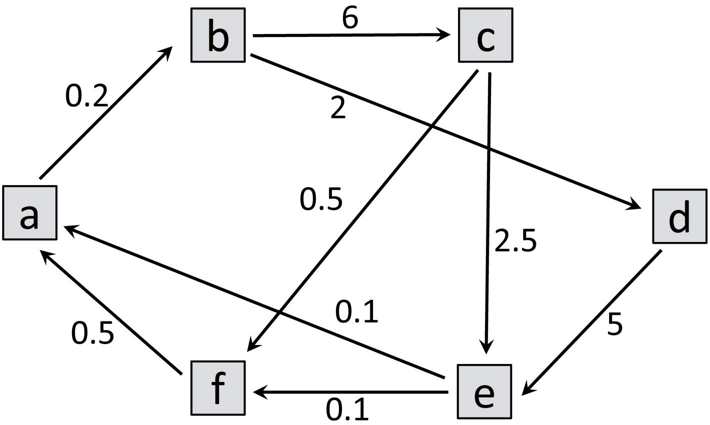
Example: Is a Pareto Improvement Possible?

$$\boldsymbol{B}(\mathbf{0}) = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}$$

Result

A Pareto improvement on the status quo $\boldsymbol{a} = \mathbf{0}$ exists if and only if $B_{12} \cdot B_{21} > 1$.

A More Complicated Example



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An interior action profile $\boldsymbol{a}$ is Pareto efficient if and only if $r(\boldsymbol{B}(\boldsymbol{a})) = 1$.

Proof Sketch: $\boldsymbol{a}^*$ Pareto-efficient $\Rightarrow r(\boldsymbol{B}(\boldsymbol{a}^*)) = 1$

Take PE $\boldsymbol{a}^*$, assume $\frac{\partial u_i}{\partial a_i}(\boldsymbol{a}^*) = -1$.

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FOC: $\forall j \qquad \displaystyle\sum_{i \neq j} \theta_i \frac{\partial u_i}{\partial a_j} - \theta_j = 0$

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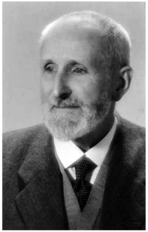

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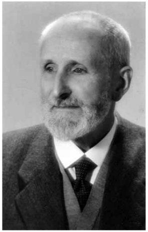
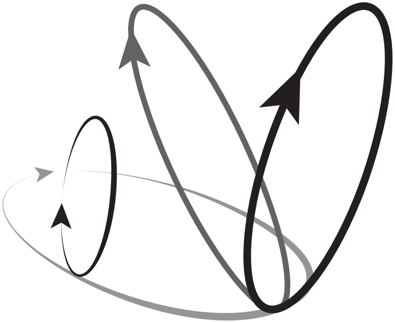


Perron-Frobenius: an eigenvalue λ of $\boldsymbol{B}$ has a nonnegative left (right) eigenvector if and only if $\lambda = r(\boldsymbol{B})$. Moreover, $\boldsymbol{B}$ has an eigenvalue $\lambda \in \mathbb{R}$ equal to $r(\boldsymbol{B})$.

Interpretation of Spectral Radius

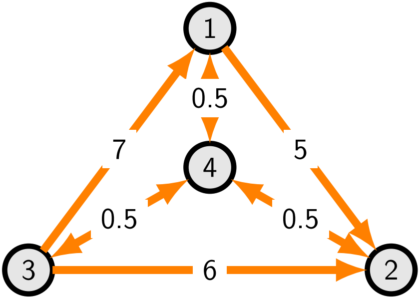
Vague Statement

The spectral radius measures the number/intensity of **cycles** in the benefits matrix.



Spectral Radius in Terms of Cycles

$$\boldsymbol{B}(\mathbf{0}) = \begin{bmatrix} 0 & 0 & 7 & 0.5 \\ 5 & 0 & 6 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}$$



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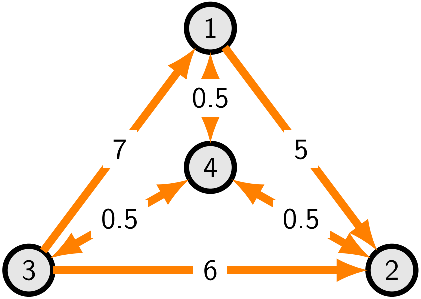

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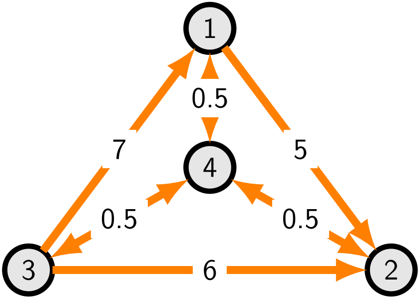

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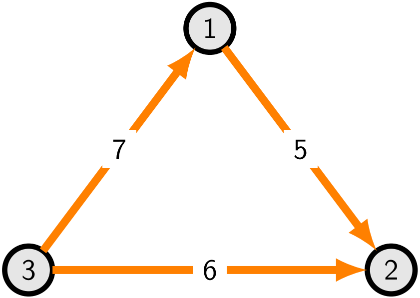

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Player 4 is essential.

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  - *cost of separation* $c_M(\boldsymbol{a}^*)$ defined as the infimum of $\sum_{i \in N} m_i(\boldsymbol{a}^*)$, taken over deviation-deterring transfers.

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Proposition

$$c_M(\mathbf{a}^*) \leq \sum \frac{\theta_i}{\theta_j} B_{ij}(\mathbf{a}^*) a_j^*,$$

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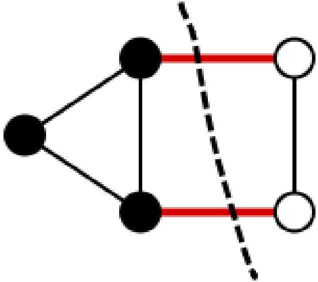

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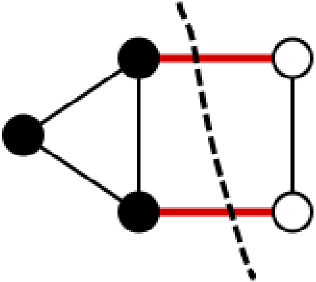

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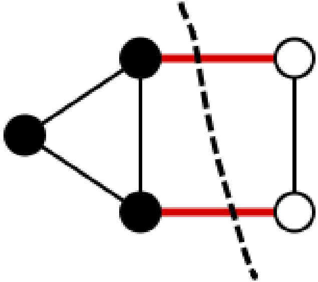

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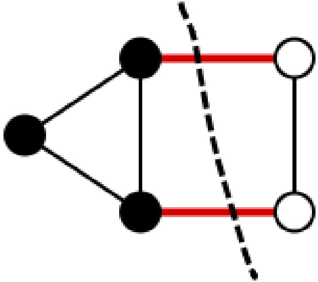

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- Small when *spectral gap* of $\mathbf{W}$ is small.

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  - Measures the returns on the best egalitarian improvement. ▸ Details

Outline

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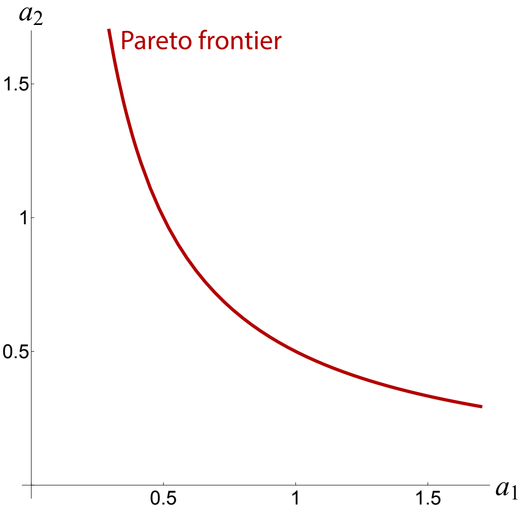

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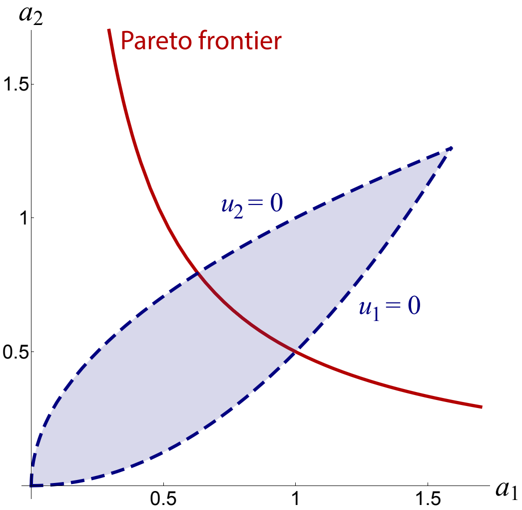

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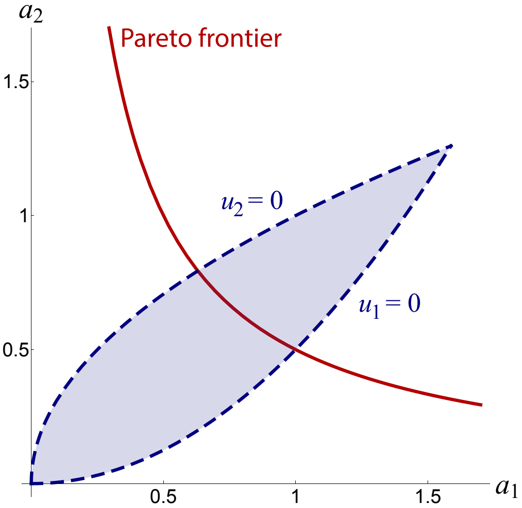

From now on, assume set of IR points is bounded.

Lindahl Outcome

Conceptually: complete the missing markets for externalities to achieve efficient provision.

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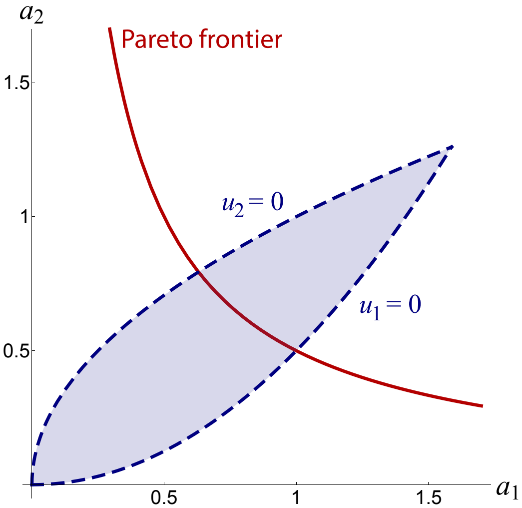
Main theorem: characterization in terms of network centrality.

Lindahl Outcome Graphically



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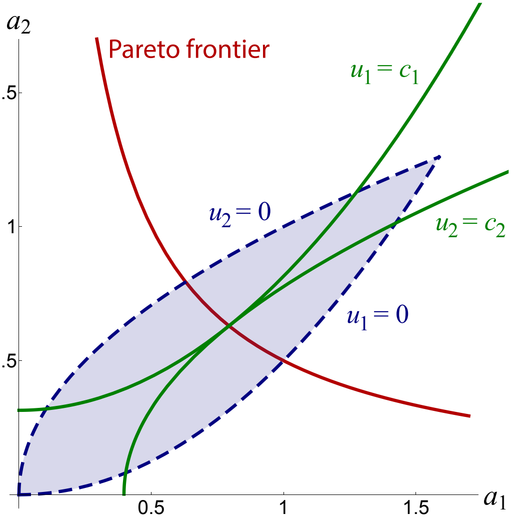

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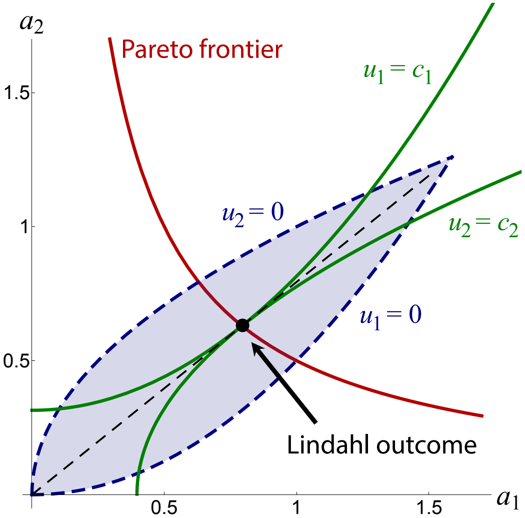

Centrality Property

Definition

$\boldsymbol{a} \in \mathbb{R}^n_+$ has *the centrality property* (or is a *centrality action profile*) if $\boldsymbol{a} \neq \mathbf{0}$ and

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- Fixed-point definition of actions.

  Agents taking high actions are those who benefit a lot (at the margin) from others who are taking high actions.

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Theorem

A nonzero $\boldsymbol{a}$ is a Lindahl outcome if and only if it has the centrality profile.

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Perron-Frobenius: an eigenvalue λ of $\boldsymbol{B}$ has a nonnegative **left** (**right**) eigenvector if and only if $\lambda = r(\boldsymbol{B})$.

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Dávila, Eeckhout, and Martinelli (JPET 09), Penta (JME 11); see also Yildiz (Games 03).

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Theorem

If $\mathbf{0}$ is inefficient and utilities are strictly concave, then: in any *efficient perfect equilibrium*, a Lindahl outcome is played.

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  To avoid equilibrium selection fight, Lindahl mechanism is the best bet.

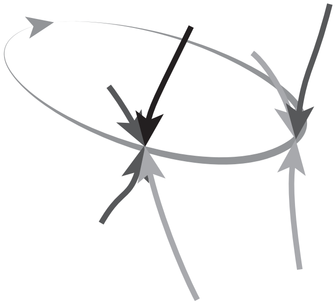
Details

Outline

Walk Interpretation of Eigenvector Centrality

Vague Statement

A node's centrality measures the number/intensity of **walks** in the benefits matrix that end at that node.

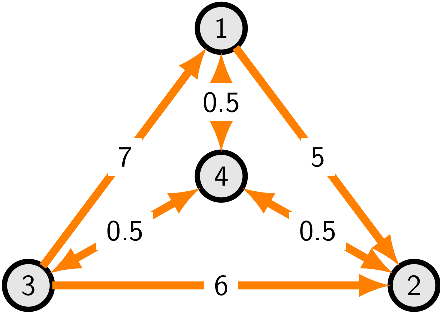


Walks and their Values

$$\boldsymbol{B}(\mathbf{0}) = \begin{bmatrix} 0 & 0 & 7 & 0.5 \\ 5 & 0 & 6 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}$$

Value of walk $w = (3, 1, 2)$:

$$\begin{aligned} v(w; \boldsymbol{B}) &= B_{13} B_{21} \\ &= 7 \cdot 5 \end{aligned}$$



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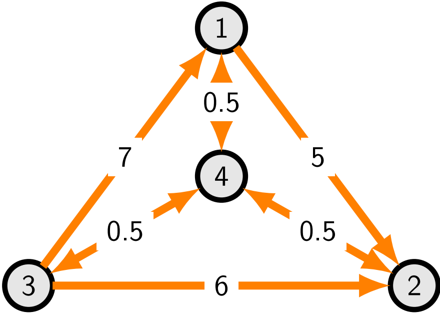
Walks can repeat nodes: e.g., $(3, 1, 2, 4, 3, 2)$.



Centrality in Terms of Walks

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Fact

Assume $\boldsymbol{B}(\boldsymbol{a})$ is aperiodic. $\boldsymbol{a}$ has the centrality property if and only if

$$\frac{a_i}{a_j} = \lim_{\ell \to \infty} \frac{V_i^{\downarrow}(\ell; \boldsymbol{B})}{V_j^{\downarrow}(\ell; \boldsymbol{B})}.$$

Each agent's effort proportional to the total value of long walks he terminates ("total incoming benefits").

Contributions

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  - Conceptual punchline: can think of market outcomes using network centrality!
  - Encouraging metaphor, but need to address "markets you can take literally".

Outline

Further Results

- Analogous characterization with transferable numeraire. Details
- Explicit formulas for centrality action profiles in parameterized economies. (New microfoundations for network centrality measures). Details
- **Next step:** analogous exercise for Walrasian outcomes in other settings to examine key nodes, robustness of market to removing nodes, etc.

Foundations for Lindahl: The Design Problem

We imagine the designer of a mechanism.

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- Given a mechanism $H = (\Sigma, g)$, let $\Sigma^*_H : \mathcal{U} \rightrightarrows \mathbb{R}^n_+$ be the equilibrium correspondence.
- Designer wants the mechanism to be **reliable**:
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  No. Has many inefficient equilibria.

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Theorem (Hurwicz 1979, Hurwicz-Maskin-Postlewaite 1994)

Recall **reliable** $=$ PE $+$ IR $+$ uhc. Assume $n \geq 3$.

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Payoff-uniqueness is achievable exactly for those $\boldsymbol{u}$ such that all Lindahl outcomes under $\boldsymbol{u}$ are payoff-equivalent. ▸ Proof of theorem

▸ Explicit condition for uniqueness ▸ Details

Literature

- Public goods.
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  - Recent applications: Brin and Page (1998); Ballester, Calvó-Armengol, and Zenou (2006); Acemoglu et al. (2012).

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- $p_j = \theta_i \cdot \frac{\partial u_i}{\partial x_j}$ where $\theta_i = \mu_i^{-1}$.

Proof of Cycles Formula for Spectral Radius

Proposition

$$r(\boldsymbol{B}) = \lim_{\ell \to \infty} \left[\sum_{\substack{c \text{ a cycle} \\ \text{of length } \leq \ell}} v(c; \boldsymbol{B}) \right]^{1/\ell}$$

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- Write $\rho = r(\boldsymbol{B})$. We have $\operatorname{trace}(\boldsymbol{B}^{\ell}) \leq n\rho^{\ell}$ always. For ℓ divisible by d, we also have $\rho^{\ell} + O(s^{\ell}) \leq \operatorname{trace}(\boldsymbol{B}^{\ell})$ with $s < \rho$.

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Write $\tau = 1 + t$ (where t is a tax). A tax of $t = r(\boldsymbol{B}(\boldsymbol{a})) - 1$ on contributions would be necessary to dissuade a social planner from increasing contributions. Back

The Spectral Radius as a Measure of Inefficiency

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A direction $\boldsymbol{d} \in \Delta$ is **egalitarian at $\boldsymbol{a}$** if every entry of $\boldsymbol{b}(\boldsymbol{a}, \boldsymbol{d})$ is the same.

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Proposition

At any $\boldsymbol{a}$, there is a unique egalitarian direction $\boldsymbol{d}^{\text{eg}}(\boldsymbol{a})$. Every entry of $\boldsymbol{b}(\boldsymbol{a}, \boldsymbol{d}^{\text{eg}}(\boldsymbol{a}))$ is equal to the spectral radius of $\boldsymbol{B}(\boldsymbol{a})$.

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- In other words, for each i,
$$\rho = \frac{\sum_i B_{ij} d_j}{d_i} = \frac{\sum_j \frac{\partial u_i}{\partial a_j} d_j}{-\frac{\partial u_i}{\partial a_i} d_i}.$$

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- By uniqueness of the Perron vector, there is no other egalitarian direction.

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Cycles Interpretation

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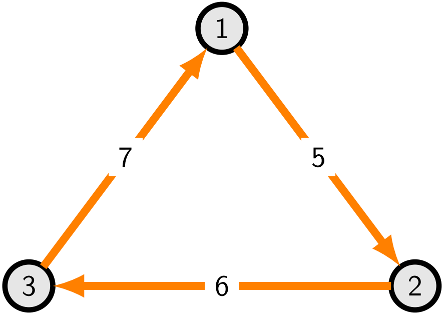

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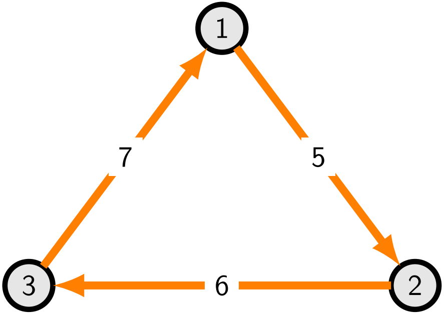

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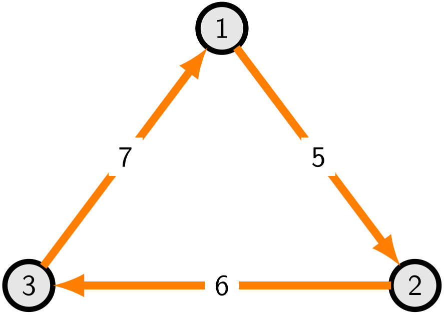

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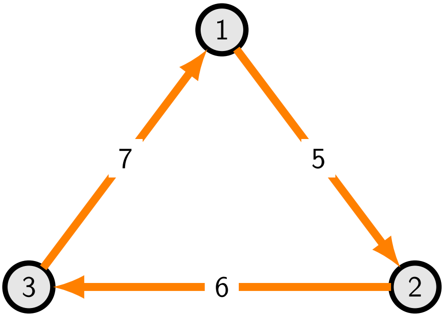

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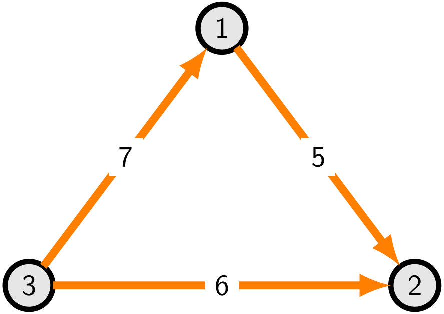

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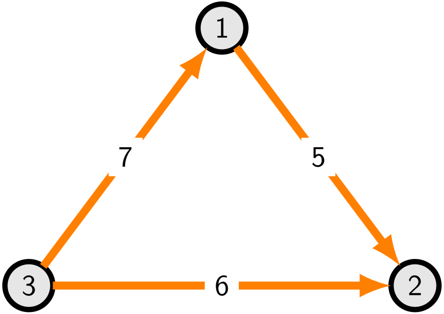

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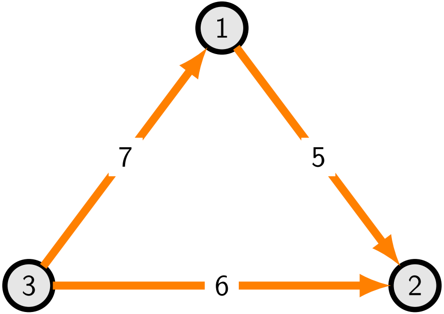

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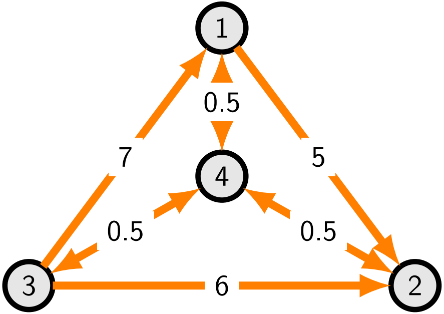

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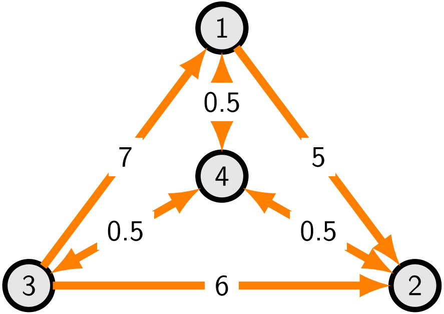
$r(\boldsymbol{B}(\mathbf{0})) > 1$

(*lots of cycles*)



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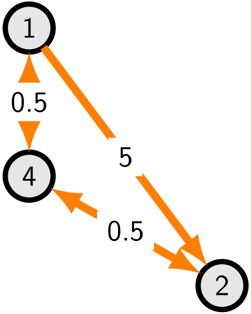

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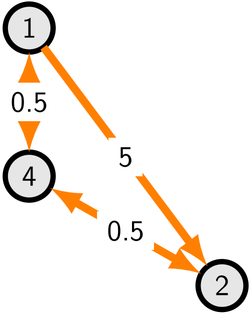
$$r(\boldsymbol{B}\left(\mathbf{0}\right)) \geq \left(5 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\right)^{1/3} > 1$$



◂ Back

Gross Substitutes

Assumption (Gross Substitutes)

Let $p_j > 0$ be the price of j's effort and 1 be i's wage. Let

$$\boldsymbol{a}^*(\boldsymbol{p}) = \operatorname*{argmax}_{\boldsymbol{a}} u_i(\boldsymbol{a}) \text{ subject to } \sum_{j \neq i} p_j a_j \leq a_i.$$

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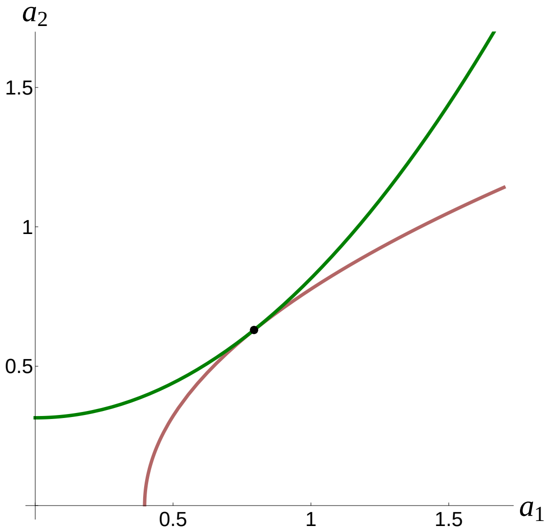
If only p_j increases, then for $k \neq i, j$, the demand a_k^* does not strictly decrease (in the strong set order); a_i^* does not strictly increase.

Back

The Proof that $L \subseteq \Sigma^*_H$ (Hurwicz, Maskin, Postlewaite)

Consider a Lindahl outcome $\boldsymbol{a}$ under preferences $\boldsymbol{u}$.



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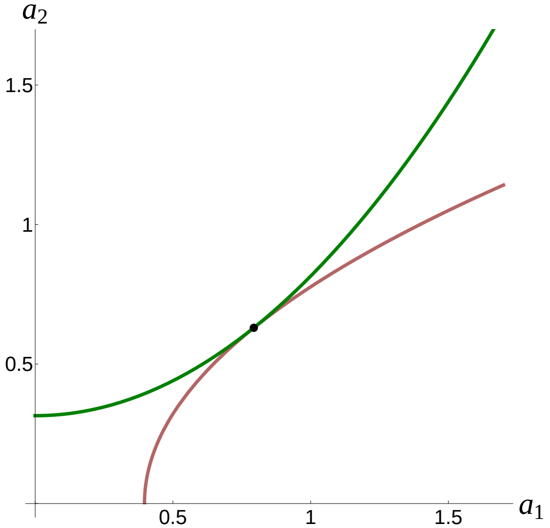

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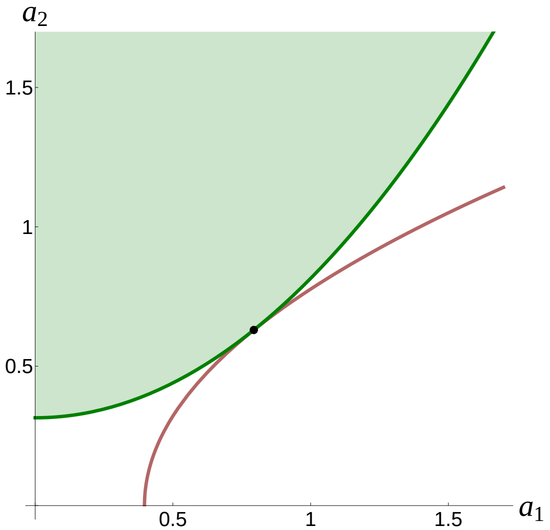

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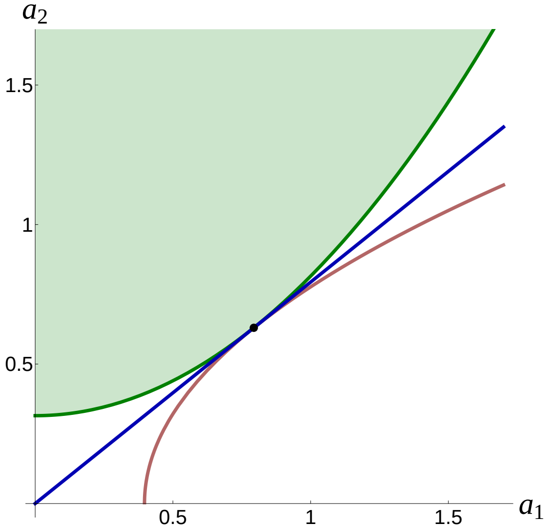

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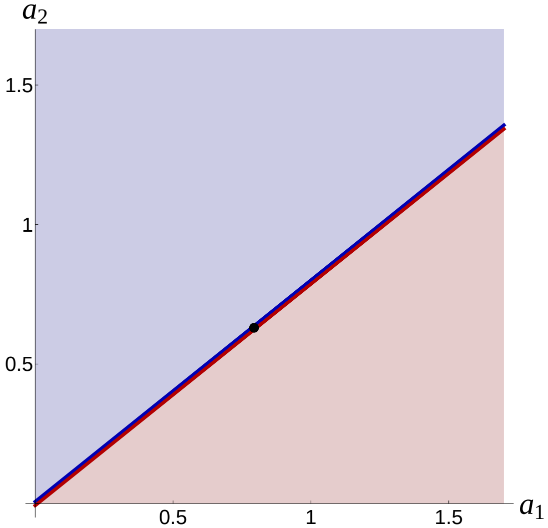

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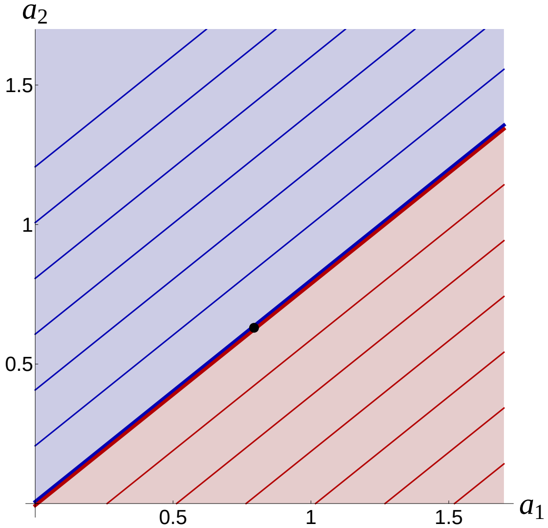

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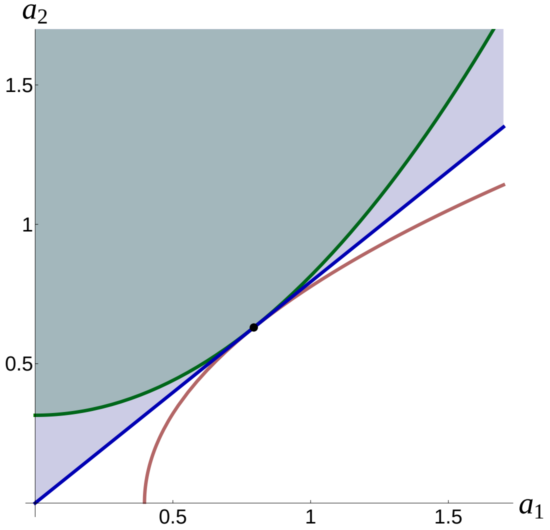
The Proof that $L \subseteq \Sigma_H^*$ (Hurwicz, Maskin, Postlewaite)

Note that each agent's "better-than-$\boldsymbol{a}$" set is strictly larger under $\widehat{\boldsymbol{u}}$ than under $\boldsymbol{u}$.

By Maskin's theorem, whatever Σ_H^* implements under $\widehat{\boldsymbol{u}}$ must also be implemented under $\boldsymbol{u}$.



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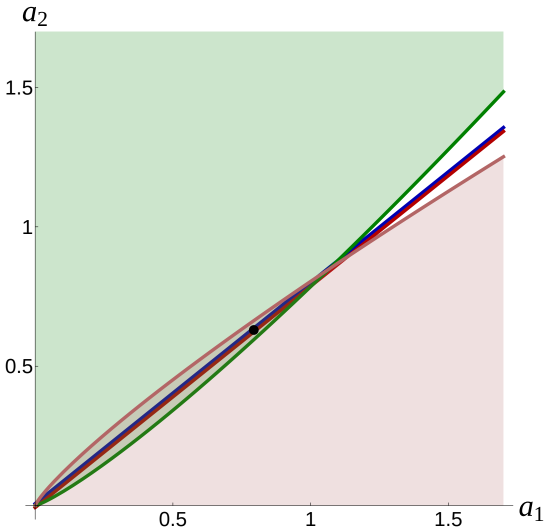
The Proof that $L \subseteq \Sigma_H^*$ (Hurwicz, Maskin, Postlewaite)

Construct preferences increasingly "near" $\widehat{\boldsymbol{u}}$ so that IR and PE alone force outcome of Σ_H^* to be near $\boldsymbol{a}$.

By continuity, $\boldsymbol{a}$ must be one of the outcomes implemented under $\widehat{\boldsymbol{u}}$.



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Proposition

The action profile $\boldsymbol{a}$ is a Lindahl outcome if and only if $\boldsymbol{\theta} = \boldsymbol{\theta B}$ where $m_i = \theta_i \left(-a_i + \sum_j B_{ij} a_j\right)$.

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Explicit Formulas: Microfoundations for Bonacich Centrality

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Say $\boldsymbol{h} = \boldsymbol{1}$. Then $a_i = \left(\begin{array}{c} \text{total value of walks} \\ \text{in } \boldsymbol{G} \text{ ending at } i \end{array} \right)$

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- Citations:
  - Yildiz (*Games* '03), Dávila and Eeckhout (*JET* '08), Dávila, Eeckhout, and Martinelli (*J Pub Econ Th* '09), **Penta** (*J Math Econ* '11).

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