Social Learning and Influence in Networks

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Plan

- Lecture notes at bengolub.net ("Current Teaching").
  - On interactive learning: rough lecture notes.
  - On sequential social learning: G and Sadler survey, Section 2.

- Quick reminder of "sacrificial lambs.

- Main material for today:
  - Finishing sequential social learning: idea of diffusion.


- Time permitting, begin linear updating models.
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Review/cleanup from yesterday:
  - Interactive learning agreement with coarse actions.

On sequential learning, MG property of likelihood ratio $r_n = q_{n+1} - q_n$.
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A social learning setting consists of:

- $\mathcal{N}$ – a set of players;
- $\mathcal{A}$ – a common action space;
- $\Theta$ – state space;
- $u : \mathcal{A} \times \Theta \rightarrow \mathbb{R}$ – a common utility function;
- $\mathcal{S}$ – private signal space;
- $\mu_i$ for each $i$ – $i$’s prior over $\Omega = \Theta \times \mathcal{S}^\mathcal{N}$.

Interactive learning protocol:

- At $t = 0$, get $s_i$ and take actions $a_i(0)$ [all simul.].
- For $t \geq 1$, observe $a_j(s)$ for all $s < t$, $j \in \mathcal{N}(n)$ and take actions $a_i(t)$ [all simultaneously].
- Let $I_{i,t}$ be the info. of $i$ at time $t$ (formally a $\sigma$-alg. on $\Omega$).
- Take a $u$ maximized by reporting $\mathbb{P}(\theta = 1 \mid I_{i,t})$.
- Players play myopic best-response to all predecessors’ strategies.
Can we say anything about agreement with a coarse action space?

If \( i \) takes action \( a \) infinitely often and \( j \) takes action \( a' \neq a \) infinitely often, then it is asymptotically common knowledge between them that \( i \) thinks \( a \) is weakly better while \( j \) thinks \( a' \) is weakly better. I.e.

\[
\mathbb{E}[u(a, \theta) - u(a', \theta) \mid \bar{I}_i] \geq 0 \quad \text{and} \quad \mathbb{E}[u(a, \theta) - u(a', \theta) \mid \bar{I}_j] \leq 0.
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Robustness of agreement: Coarse actions

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- By the Agreeing to Disagree argument, this can only happen if both are equalities. So neighbors have to be indifferent between actions taken infinitely often.
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nature draws $\theta \in \{0, 1\}$ with $q_0 = \mathbb{P}(\theta = 1)$.

Mr. $n \in \mathbb{N}$ acts at time $n$

sees $I_n$:
- signal $s_n \in S$;
- actions of nbhd $\mathcal{N}(n)$; dist $\mathbb{Q}$ is joint of $(\mathcal{N}(n))_n$

makes choice $a_n \in \{0, 1\}$

About signals:
- conditionally i.i.d given $\theta$, $s_n \sim \mathbb{F}_\theta$
- $\mathbb{F}_0 \neq \mathbb{F}_1$. 
Recall: sequential social learning model, $J_n$ consists of all actions before $n$, and $q_n = \mathbb{P}[\theta = 1 \mid J_n]$.

$r_n = \frac{q_n}{1 - q_n}$.

Conditional on $\theta = 0$, $(r_n)_n$ is a martingale.
Sequential social learning using (conditional) martingale

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- \( r_n = \frac{q_n}{1-q_n} \).

- Conditional on \( \theta = 0 \), \((r_n)_n\) is a martingale.

\[
\frac{q_{n+1}}{1 - q_{n+1}} = \frac{q_n}{1 - q_n} \times \frac{\mathbb{P}[a_n \mid q_n, \theta = 1]}{\mathbb{P}[a_n \mid q_n, \theta = 0]}
\]

\[
\mathbb{E}\left[\frac{q_{n+1}}{1 - q_{n+1}} \mid r_n, \theta = 0\right] = \frac{q_n}{1 - q_n} \times \mathbb{E}\left[\frac{\mathbb{P}[a_n \mid r_n, \theta = 1]}{\mathbb{P}[a_n \mid r_n, \theta = 0]} \mid r_n, \theta = 0\right].
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$$\mathbb{E} \left[ \frac{q_{n+1}}{1-q_{n+1}} \mid r_n, \theta = 0 \right] = \frac{q_n}{1-q_n} \times \mathbb{E} \left[ \frac{\mathbb{P}[a_n \mid r_n, \theta = 1]}{\mathbb{P}[a_n \mid r_n, \theta = 0]} \mid r_n, \theta = 0 \right].$$

$$\sum_{a_n} \frac{\mathbb{P}[a_n \mid r_n, \theta = 1]}{\mathbb{P}[a_n \mid \circ]} \mathbb{P}[a_n \mid \circ] = \sum_{a_n} \mathbb{P}[a_n \mid r_n, \theta = 1] = 1$$
Suppose there is a deterministic sequence \( (m_k)_{k=1}^{\infty} \) of “candidate sacrificial lambs.”

Their neighborhoods are, independently, empty with some probability \( e_k \), such that \( \sum_k e_k = \infty \).

For each \( k \), for any \( \epsilon \) there is an \( N(\epsilon) \) large enough so that any agent \( n \geq N(\epsilon) \) observes \( m_k \) with probability at least \( 1 - \epsilon \).
SSLM setup reminder

- nature draws $\theta \in \{0, 1\}$ with $q_0 = \mathbb{P}(\theta = 1)$.
- Mr. $n \in \mathbb{N}$ acts at time $n$
- sees $I_n$:
  - signal $s_n \in S$;
  - actions of nbhd $\mathcal{N}(n)$ -- dist $\mathcal{Q}$ is joint of $(\mathcal{N}(n))_n$
- makes choice $a_n \in \{0, 1\}$

About signals:
- conditionally i.i.d given $\theta$, $s_n \sim \mathbb{F}_\theta$
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Some definitions:
- $J_n$ consists of all actions before $n$, dist $\mathcal{Q}$ is joint of $(\mathcal{N}(n))_n$.
- $p_n = \mathbb{P}[\theta = 1 \mid s_n]$.

Let $\underline{\beta}$ be the infimum of support of $p_n$, and $\overline{\beta}$ be the supremum.
Fact

There is a distribution $\tilde{s}$, called the expert signal, with support $\{s, \bar{s}\}$ s.t.

$$P(\theta = 1 | \tilde{s} = s) = \beta \quad P(\theta = 1 | \tilde{s} = \bar{s}) = \bar{\beta}$$
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Diffusion occurs in an eqm if

\[
\lim_{n \to \infty} u(a_n, \theta) \geq \text{payoff under expert signal}
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Diffusion: A new concept

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Diffusion occurs under $Q$ if it occurs in every equilibrium, for every signal distribution.
Fact

There is a distribution \( \tilde{s} \), called the expert signal, with support \( \{s, \bar{s}\} \) s.t.

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\begin{align*}
P(\theta = 1 | \tilde{s} = s) &= \beta \\
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Two famous notions:

- **herd**: a.s. \( (a_n)_n \) converges;
- **information cascade**: a.s. players eventually ignore private information.