Social Learning and Influence in Networks

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  - On interactive learning: rough lecture notes.
  - On sequential social learning: G and Sadler survey, Section 2.

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  - "Social Learning Equilibria," Manuel-Frank, Mossel, Tamuz, and Sly (2018).

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  - Time permitting, begin linear updating models.

Social learning setting and interactive learning protocol

A social learning setting consists of:

- N – a set of players;
- A – a common action space;
- Θ – state space;
- $u : A \times \Theta \to \mathbb{R}$ – a common utility function;
- S – private signal space;
- μ_i for each i – i's prior over $\Omega = \Theta \times S^N$.

Interactive learning protocol:

- At $t = 0$, get s_i and take actions $a_i(0)$ [all simul.].
- For $t \geq 1$, observe $a_j(s)$ for all $s < t$, $j \in \mathcal{N}(n)$ and take actions $a_i(t)$ [all simultaneously].
- Let $I_{i,t}$ be the info. of i at time t (formally a σ-alg. on Ω).
- Take a u maximized by reporting $\mathbb{P}(\theta = 1 \mid I_{i,t})$.
- Players play myopic best-response to all predecessors' strategies.

Robustness of agreement: Coarse actions

- Can we say anything about agreement with a coarse action space?
- If i takes action a infinitely often and j takes action $a' \neq a$ infinitely often, then it is asymptotically common knowledge between them that i thinks a is weakly better while j thinks a' is weakly better. I.e.

$$\mathbb{E}[u(a,\theta) - u(a',\theta) \mid \overline{I}_i] \geq 0 \qquad \text{and} \qquad \mathbb{E}[u(a,\theta) - u(a',\theta) \mid \overline{I}_j] \leq 0.$$

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- Might hope it's nongeneric, but probabilities endogenous.
- Gale and Kariv (2003), Rosenberg, Solan, and Vieille (2011), Mueller-Frank (2013)

SSLM setup reminder

- nature draws $\theta \in \{0,1\}$ with $q_0 = \mathbb{P}(\theta = 1)$.
- Mr. $n \in \mathbb{N}$ acts at time n
- sees I_n:
  - signal $s_n \in S$;
  - actions of nbhd $\mathcal{N}(n)$; dist $\mathbb{Q}$ is joint of $(\mathcal{N}(n))_n$
- makes choice $a_n \in \{0,1\}$

About signals:

- conditionally i.i.d given θ, $\quad s_n \sim \mathbb{F}_\theta$
- $\mathbb{F}_0 \neq \mathbb{F}_1$.

Sequential social learning using (conditional) martingale

- Recall: sequential social learning model, J_n consists of all actions before n, and $q_n = \mathbb{P}[\theta = 1 \mid J_n]$.
- $r_n = \frac{q_n}{1-q_n}$.
- Conditional on $\theta = 0$, $(r_n)_n$ is a martingale.

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$$\frac{q_{n+1}}{1-q_{n+1}} = \frac{q_n}{1-q_n} \times \frac{\mathbb{P}[a_n \mid q_n, \theta = 1]}{\mathbb{P}[a_n \mid q_n, \theta = 0]}$$

$$\mathbb{E}\left[\frac{q_{n+1}}{1-q_{n+1}} \mid r_n, \theta = 0\right] = \frac{q_n}{1-q_n} \times \mathbb{E}\left[\frac{\mathbb{P}[a_n \mid r_n, \theta = 1]}{\mathbb{P}[a_n \mid r_n, \theta = 0]} \mid r_n, \theta = 0\right].$$

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$$\sum_{a_n} \frac{\mathbb{P}[a_n \mid r_n, \theta = 1]}{\mathbb{P}[a_n \mid \circ]} \mathbb{P}[a_n \mid \circ] = \sum_{a_n} \mathbb{P}[a_n \mid r_n, \theta = 1] = 1$$

Sacrifical lambs

- Suppose there is a deterministic sequence $(m_k)_{k=1}^{\infty}$ of "candidate sacrificial lambs."
- Their neighborhoods are, independently, empty with some probability e_k, such that $\sum_k e_k = \infty$.
- For each k, for any ϵ there is an $N(\epsilon)$ large enough so that any agent $n \geq N(\epsilon)$ observes m_k with probability at least $1 - \epsilon$.

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Some definitions:

- J_n consists of all actions before n, dist $\mathbb{Q}$ is joint of $(\mathcal{N}(n))_n$.
- $p_n = \mathbb{P}[\theta = 1 \mid s_n]$.

Let $\underline{\beta}$ be the infimum of support of p_n, and $\overline{\beta}$ be the supremum.

Diffusion: A new concept

Fact

There is a distribution $\widetilde{s}$, called the expert signal, with support $\{\underline{s}, \overline{s}\}$ s.t.

$$\mathbb{P}(\theta = 1 \mid \widetilde{s} = \underline{s}) = \underline{\beta} \qquad \mathbb{P}(\theta = 1 \mid \widetilde{s} = \overline{s}) = \overline{\beta}$$

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Two famous notions:

- **herd**: a.s. $(a_n)_n$ converges;
- **information cascade**: a.s. players eventually ignore private information.